

INSTRUCTION IN ARITHMETIC

Twenty-Fifth Yearbook

**THE NATIONAL COUNCIL OF TEACHERS
OF MATHEMATICS**

Washington, D. C., 1960

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PREFACE

INSTRUCTION IN ARITHMETIC has been prepared primarily to serve the needs of teachers of arithmetic, principals of and supervisors in the elementary school, and students in elementary education. This book presents a report of progress that has occurred in the field of arithmetic since the publication of the last yearbook in this subject. The Yearbook Committee hopes that this publication will enable the teacher who may be unfamiliar with "modern mathematics" to understand something about this phase of mathematics and how it is certain to affect the teaching of arithmetic.

Study Groups at various universities and different publications of the National Council of Teachers of Mathematics and of other national organizations have stressed the need and place of modern mathematics in secondary schools and colleges. Unfortunately, very little attention has been given by these groups to needed changes in arithmetic. Suggestions for improving instruction in arithmetic as given in this yearbook should help the teacher to meet the needs of a modern program in beginning elementary mathematics.

The Yearbook Committee is grateful to the members of the Reviewing Committee who read the manuscript carefully and offered constructive criticism for its revision. Members of the Reviewing Committee are: William A. Brownell, Leo J. Brueckner, Guy T. Buswell, John R. Clark, and Robert L. Morton.

Walter A. Graves of the *NEA Journal* was most helpful in making available to us the photographic files of the *Journal*. Mary L. Grau of the Montgomery County, Maryland, schools supplied the photograph on page 137. The Bell Aircraft Corporation provided the photograph on page 38.

The Yearbook Committee expresses its thanks to Mercedes S. Grossnickle for her assistance throughout the project and particularly for preparing the index.

The Committee

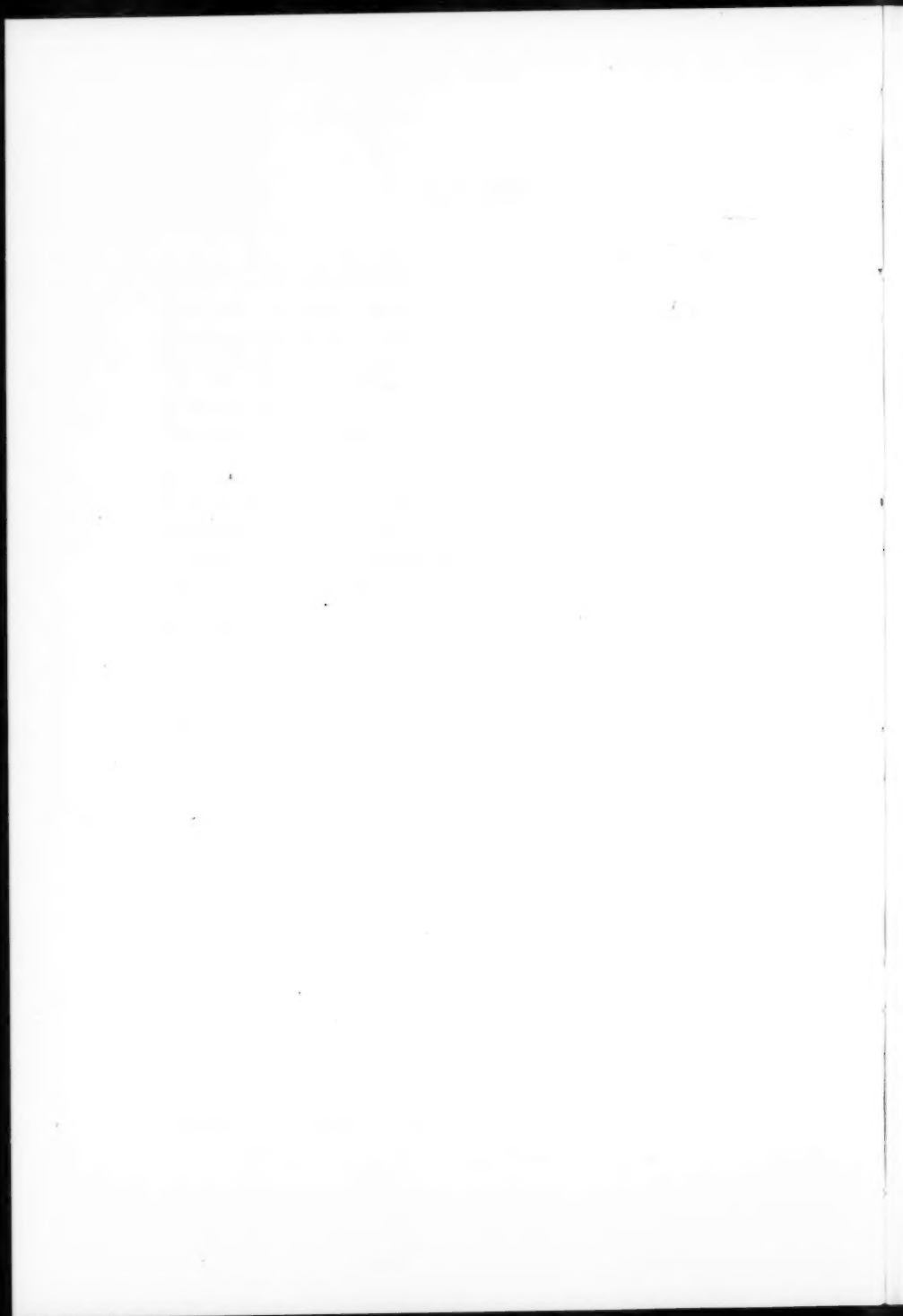
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Introduction

FOSTER E. GROSSNICKLE

INSTRUCTION IN ARITHMETIC is the third yearbook published by the National Council of Teachers of Mathematics which deals with arithmetic. Earlier volumes in this field were the Tenth (1935) and the Sixteenth (1941) Yearbooks. Brownell (1)*, in the former, characterized the difference between learning based on meaning and understanding and rote learning resulting from repetition. In the same book, Wheeler (8) emphasized the view that learning is not a matter of forming bonds so much as it is a process of integrating experiences. He suggested that more emphasis be placed on syntheses and wholes and less on analyses and parts. He so heavily emphasized the role of Gestalt in learning that he considered drill unnecessary. This book supported a theory of learning arithmetic different from the so-called atomistic theory predominantly advocated in 1930 by the Twenty-Ninth Yearbook of the National Society for the Study of Education (6).

The Tenth Yearbook also stressed the informational phase of

* A number in parentheses (x) indicates a reference at the end of the chapter; ($x:y$) is used to refer to page y of reference x .

arithmetic and emphasized a wider concept of its nature than the view then prevalent that arithmetic is predominantly computational. This publication may be considered significant because it challenged the atomistic theory of learning. It did not accept the narrow concept of development of skill in computation as the exclusive goal to be attained from the study of arithmetic.

The Sixteenth Yearbook made a distinct contribution to the teaching of arithmetic in three ways. First, McConnell (3) showed that organization and structure are important elements in learning and stressed the fact that learning is a meaningful process. He recognized the value of drill and the necessity of a period of fixation of learnings. This view is now generally accepted in contrast to the view expressed by Wheeler. (While both men embrace the same general theories of learning, McConnell's view is more in line with modern thinking than that of Wheeler.) Second, Sauble (7) gave valuable suggestions for implementing a program which gives emphasis to teaching of meanings and developing understandings. She showed how to make use of certain supplementary aids and materials in the teaching of arithmetic. Third, the yearbook emphasized that there are two phases of arithmetic: the social phase or the significance of arithmetic as it is applied in our daily living, and the phase pertaining to the relationships which exist among numbers or quantities. An adequate program in arithmetic recognizes each of these phases of the subject.

The emphasis given predominantly to either the social phase or the mathematical phase of arithmetic frequently resulted in a dichotomy. Some teachers gave considerable emphasis to the social phase; others, to the mathematical phase. Such a dualism keeps arithmetic from becoming a unified subject. Each phase should be an integral part of arithmetic and the teacher should not try to isolate one phase from the other in presenting the subject to the learner.

Instruction in Arithmetic subscribes to the field theories of learning as expressed in the Fiftieth Yearbook of the National Society for the Study of Education (2:143-54). The field psychologist looks upon the structured whole of arithmetic rather than upon the elements of its various unrelated parts. The theory

of learning as expressed by Wheeler (8) and McConnell (3) represents the view of a field psychologist. Meaning and understanding are vital elements in enabling the pupil to grasp and retain what he learns. The Yearbook Committee faced the problem of implementing a program that would stress meaning and understanding in the teaching of arithmetic.

This yearbook emphasizes three important elements in an effective program dealing with arithmetic. First, the nature of the subject is such that it has a cultural value; it is structured; properly taught, it leads to a unique, quantitative way of thinking; and it is basic in the further study of mathematics. Second, a program for the learning of arithmetic should recognize such factors as the mental hygiene of the classroom, adequate records for guidance, provision for optimum individual growth, use of materials, and ability to read quantitative statements. Third, a specific course in background mathematics is recommended as essential in the training of teachers of arithmetic.

The three phases of the program mentioned indicate that *Instruction in Arithmetic* represents a progress report in clarifying ideas that have already been formulated. It is almost certain that classroom practice has not caught up with the recommendations of the two arithmetic yearbooks mentioned. A major objective of the yearbook is to help administrators and teachers understand that the in-service training of teachers as well as the preparation of prospective teachers will necessitate a great change from many current practices. Much consideration must be given to such topics as an understanding of the structure of arithmetic, mental hygiene in the classroom, guidance, provision for individual differences, use and preparation of materials, and other vital related topics bearing on the program for effective teaching of arithmetic. Our discussion of these topics aims to implement such a program.

PLAN OF THE YEARBOOK

The committee responsible for the Twenty-Fifth Yearbook made a detailed outline of each chapter before attempting to secure writers for the chapters. The writers were then asked to follow their outlines as closely as possible. This plan was used in

order to keep to a minimum duplication of material. A certain amount of duplication in the development of related topics is essential, but a great amount of duplication is not defensible.

There are noticeable omissions of two topics that usually appear in arithmetic yearbooks—evaluation and curriculum. These omissions are the result of a decision made by the planning committee. A forthcoming yearbook of the National Council of Teachers of Mathematics will deal with the problem of evaluation, and the committee assumed that a much more comprehensive treatment would be given the topic at that time than could possibly be given in a current chapter.

With the exception of a chapter dealing with kindergarten and grades 1 and 2, the subject of curriculum receives no consideration in this book. The Council has a committee working on the curriculum in arithmetic. Although the work of this committee will not appear in a yearbook, their published recommendations should deal more adequately with problems pertaining to curriculum than a chapter we might have included. A program for kindergarten and grades 1 and 2 is included because this is one of the least explored areas in the field of arithmetic.

BRIEF SYNOPSIS OF THE YEARBOOK

The chapters of this book may be divided or grouped into five parts. Part one deals with the cultural value of arithmetic, its structure, learnings peculiar to this subject, and its content. Chapters 2-5 constitute this segment of the book. Part two deals with the learner and factors affecting him in learning arithmetic. Chapters 6-10 comprise this phase of the subject. Part three deals predominantly with the meaning of modern mathematics and its impact on arithmetic. Chapters 11 and 12 deal with this area. Parts four and five consist of one chapter each. Part four deals with the training of teachers of arithmetic. Part five gives a brief summary of investigations and other publications dealing with arithmetic since the presentation of the last yearbook in arithmetic by the Council in 1941.

The Nature of Arithmetic

Part one contains chapters written by Welmers, Van Engen and Gibb, Clark, and Spitzer. Welmers gives a brief summary of the

contributions made by different peoples to the development of arithmetic from its beginning in counting to its use in the electronic computer. He points out the role arithmetic plays in our daily lives, in business, in government, and in science and technology. Our cultural pattern today may be characterized as being largely scientific. As science modifies our environment, arithmetic is the alphabet of the mathematics which enables us to understand our culture.

Van Engen and Gibb emphasize the structure of arithmetic and the principles or laws governing its operations. These laws have designated names in the field of mathematics and govern the operations within the structure of arithmetic. The term *structure* implies the same function in the system of numeration as it does in construction of buildings.

Clark formulates a list of basic generalizations in structural arithmetic, in geometry, and in social applications of arithmetic. He wisely shows the teacher how to lead the pupil to discover and make these generalizations for himself. Clark stresses the fact that the mere acquisition of generalizations is not important. The chief value comes from the thinking the pupil must do in order to make a particular generalization.

Spitzer shows the great variability among current programs in arithmetic for kindergarten and grades 1 and 2. He recommends that the work for the first grade and the first half of the second grade be planned, but that no emphasis be given to mastery of the basic facts during this period. Beginning the second semester of the second grade, the program in arithmetic should be directed towards mastery of all the basic facts in addition and subtraction. The absence of scientific data to substantiate Spitzer's recommendations doubtless will cause a diversity of opinion regarding some of his proposals.

Factors Affecting the Learning Process

Part two deals with the learner and factors affecting him. The writers of the five chapters comprising this section are Jones and Pingry, Sauble and Thiele, Dickey and Taylor, Spencer and Russell, and Hardgrove and Sultz.

One of the major problems in instruction is the task of guiding

pupils in areas of learning where success will be almost certain. Then the teacher must provide the kinds of experiences which will enable the pupil to develop according to his capacity to learn. Jones and Pingry show how to utilize both administrative procedures and differentiation of curriculum in order to provide for the great range of differences in a class. They discuss such controversial problems as homogeneous grouping and acceleration, and give their views concerning steps to take in dealing with these issues. Sauble and Thiele describe a program for providing essential records to enable the teacher to give adequate guidance in arithmetic. This phase of the teaching and learning program should begin when the pupil enters school and should not be delayed until he enters junior or senior high school.

Dickey and Taylor show how attitudes and other intangibles in teaching are dynamic factors in learning any subject, especially arithmetic. The authors formulate certain principles for good mental health in the classroom and describe some of the factors which affect mental health. They point out activities for the teacher which contribute to mental health. These writers do not confine their discussion to the narrow field of the classroom, but they include various agencies in the pupil's environment which affect the learning process, such as the home, church, and school.

Reading is a cause of difficulty, especially in problem solving. Spencer and Russell show that reading in arithmetic may be classified as either primary or secondary. Primary reading consists in reading about objects or processes. Secondary reading applies to reading of signs or symbols used to represent objects or relationships. *Mathematicking* as an application of the reading process is needed to give mathematics meaning and significance. Reading in arithmetic is a highly specialized form of reading. The authors outline a plan so that the teacher can help the pupil read to understand statements which express quantitative relationships as used in verbal problems.

The use and kinds of supplementary aids for teaching arithmetic have increased rapidly within the past decade. The great variety of aids and gadgets available for equipping the arithmetic classroom make it imperative that careful scrutiny be applied to them. Hardgrove and Sultz suggest criteria for the selection and

use of these aids. They also list some of the most effective materials for supplementing the use of the textbook. Teachers of arithmetic should give careful consideration to the plan they propose.

Terminology in Arithmetic

Part three deals mainly with the impact of modern mathematics on arithmetic. Boyer, Brumfiel, and Higgins give a glossary of arithmetical terms. This list may be divided into three parts. The first part contains definitions of many widely used terms. The second covers the often misused terms found in dealing with measurement, such as accuracy, precision, relative error, and significant digits. The third part contains terms that are unfamiliar to most teachers of arithmetic—those dealing with set theory. They are included due to the great change taking place in the reorganization of secondary mathematics, a change that is certain to produce a noticeable impact in arithmetic. The Yearbook Committee hopes that the list of terms given in this chapter may be of great help in establishing standard definitions of terms and concepts in arithmetic. The present list is merely a beginning in achieving that desirable goal.

Swain discusses the meaning of the set theory as it applies to arithmetic. He does not attempt to define a set, as it represents a primitive concept. He emphasizes the importance of the interplay between the general and the particular, how to teach both the cardinal and ordinal concepts in counting, and how to interpret the processes in terms of set terminology. The reader need not be versed in modern mathematics to get an understanding of set theory and its application to learning and teaching arithmetic.

Teacher Training in Arithmetic

Part four discusses the preparation of the teacher of arithmetic. The success of any program in this field depends upon the teacher, whose background and understanding in arithmetic are vital factors in the presentation of the subject. Ruddell, Dutton, and Reckzeh made a questionnaire study of current programs offered in teachers colleges, with specific reference to the contents of courses in background mathematics for teachers of arithmetic. In

light of their findings the authors propose a minimum course of background mathematics, emphasizing the structure of arithmetic, to enable the teacher to understand the subject. The Yearbook Committee fully endorses the recommendations in this report as constituting the minimum amount of mathematical preparation for teachers of arithmetic.

Research in Arithmetic

Part five presents investigations in arithmetic since the publication in 1941 of the Sixteenth Yearbook (4), which contains a list of 100 selected research articles in arithmetic. Since then no yearbook has compiled the newer investigations in this subject. Schaaf has now brought the list up to date. A task of this kind demands fine discrimination on the part of the compiler, as he must be the sole judge in determining whether or not a study is to be reported. Schaaf has made an excellent selection of studies in the field of elementary mathematics. Students wishing to investigate a particular topic in arithmetic should find Schaaf's list of approximately 200 annotated references of vital significance.

From the brief resume above, it is seen that this yearbook holds to the view that the pupil should be aware of the cultural value of arithmetic and its structure, that the teacher should guide and counsel the pupil from the first grade throughout his study of arithmetic so that profitable experiences can be provided to challenge his ability, and that the teacher should have the kind of academic background in the subject that will enable him to understand and implement a modern program in arithmetic.

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Arithmetic in Today's Culture

EVERETT T. WELMERS

THE EXACT WAY in which men first began to communicate with one another is lost in prehistoric mists. We assume, however, that within small groups specific sounds made by individuals began to be associated with certain objects. When sounds representing action or description were added, the transfer of more information became possible. As these small groups came together for convenience or protection and in conflict, languages began to take form.

THE LANGUAGE OF QUANTITY

Hidden among the fundamentals of communication were the roots of the numerical process. A simple object name, for example, *stone* or *wolf*, at first conveyed much of the needed information. Some distinction between *one* and *many* was perhaps the next essential. A repetition of the object name (for example *stone, stone*) may have indicated more than one before the word itself was modified (*stones*). The ancient Greeks carefully distinguished

among one, two, and many by use of singular, dual, and plural for nouns and verbs.

It is plausible to assume that in the primitive environment, a single item was compared in some way to the individual. For example, a man tracked *one* animal, fought a *single* enemy, warmed himself by *a* fire. This identification of the singular is possible without much abstraction. If such a singular correspondence could not be made, the individual recognized that a plural was involved.

The concept of cardinality, the common property possessed by groups whose members can be put into one-to-one correspondence with each other, was necessary to progress in communication. Why should *three wolves*, *three trees*, and *three people* have the same numerical adjective *three* assigned to them? Because a wolf or a tree can be associated with each person, and one person with each wolf or tree. The use of an individual's fingers may have been the key by which the transfer was made; three wolves correspond to three fingers, as do three people. The property of *three-ness* can then be dissociated from an object simulation, the three fingers at the end of an arm, and conceived as an abstract form. Finally, a symbol such as III or 3 can be used to designate this property or quantity. The use of *digit* to designate a number as well as a finger or toe and methods of counting by fives or tens (related to the number of digits available) are traces of early attempts at counting on the fingers and toes.

Only when the cardinal abstraction is understood can arithmetical ideas be correctly transmitted. For purely communicative purposes the elementary arithmetic operations need not be highly developed. "How many more?"—the question that leads to subtraction—may be the most natural step into the simple operations. Applications of number concepts to the requirements of living were essential to the first formulations of the addition, subtraction, multiplication, and division processes.

THE LANGUAGE OF SIZE

The use of integers to indicate the number of objects is relatively straightforward compared with the application of integers

in the representation of size. Fundamental to the size concept is the idea of units. Although the basis of choice of such units forms an interesting study, a more significant problem arithmetically is manipulation with multiples of such units. The combination of three unit lengths and two unit lengths to give five unit lengths seems close to reality and furnished an early tie between arithmetic and geometry. The size of a unit length is arbitrary, but it should be appropriate to the problem. A micron unit (one thousandth of a millimeter) is not appropriate to stellar distances, nor is the light year a suitable unit when applied to precision machining.

No matter how small a unit is chosen, it is not always possible exactly to represent a particular distance, area, or volume by an integral number of units. Although the fraction $\frac{1}{4}$ ft. can be interpreted as an integral 3 inches by a change of units, the $\sqrt{2}$ can never be exactly expressed as an integer no matter how small the units may be. This means that there are no two integers whose ratio is $\sqrt{2}$; therefore $\sqrt{2}$ is called an irrational number.

There are many interesting relationships, as well as noticeable absences of relationships, among various measures of size. Usually a closer connection exists between linear measures and measures of area than between linear measures and measures of volume. The sizes of baskets or jars used by the ancients for determining volume seem to have had no connection either with the units of length or with the number of objects which they would hold. Only in modern times has a consistent distance-volume relationship been established, as in the metric system.

The application of cardinal numbers to practical problems of size required a generalization to rational and irrational numbers. Theoretical justification of such numbers has been a major mathematical problem since the golden age of Greek philosophy and has only been satisfactorily answered in modern times.

HISTORY OF OUR NUMBER SYSTEM

The position of a digit in numerical representation is fundamental to our arithmetic. This was not the case, however, in all civilizations. Roman numerals represent a system in which position has little significance: III illustrates repetition of a unit to repre-

sent three; IX, subtraction of a unit from ten to give nine; and L, C, and M, the use of single letters to represent large quantities. In actual numerical manipulations both Greeks and Romans frequently used counting boards in which some positional significance was implied. The abacus and soroban were definitely based on positional concepts.

Bases of Number Systems

The use of ten as the base for our number system is so deeply ingrained that we are frequently thrown off balance when other possibilities are discussed. In our modern numerical concepts, however, there are many applications of other bases.

If we call B the *base* of our number system, then every whole number can be expressed in the form

$$a_n B^n + a_{n-1} B^{n-1} + \cdots + a_1 B^1 + a_0 B^0.$$

Thus if B is ten,

$$\begin{aligned} 4027 &= 4 \times 10^3 + 0 \times 10^2 + 2 \times 10^1 + 7 \times 10^0 \\ &= 4000 + 0 + 20 + 7. \end{aligned}$$

If B is eight, the numeral 4027 represents a different number, the value of which in decimal notation is

$$\begin{aligned} &4 \times 8^3 + 0 \times 8^2 + 2 \times 8^1 + 7 \times 8^0 \\ &= 2048 + 0 + 16 + 7 \\ &= 2071. \end{aligned}$$

If negative powers of B are permitted, we can express fractions, such as

$$\begin{aligned} 642.305 &= 6 \times 10^2 + 4 \times 10^1 + 2 \times 10^0 + 3 \times 10^{-1} \\ &\quad + 0 \times 10^{-2} + 5 \times 10^{-3} \\ &= 600 + 40 + 2 + \frac{3}{10} + \frac{0}{100} + \frac{5}{1000}. \end{aligned}$$

If B is two, 110.101 equals in decimal notation

$$\begin{aligned} 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 + 1 \times 2^{-1} + 0 \times 2^{-2} + 1 \times 2^{-3} \\ &= 4 + 2 + 0 + \frac{1}{2} + 0 + \frac{1}{8} \\ &= 6\frac{5}{8}. \end{aligned}$$

Our modern decimal system, with base ten, has been heavily influenced by the use of ten fingers on human hands. It is only natural that the five fingers of one hand might influence a preference for five in some notations. The abacus, still used by millions today, has five as a secondary base. In certain electronic computers the punched cards used as inputs also show the base five subsidiary to the decimal positional concepts.

Traces of other bases exist in our language today in spite of the dominance of the decimal notation. "Four score and seven years ago" illustrates the use of base twenty in the word *score*; here $4 \times 20^1 + 7 \times 20^0$ is used to represent the decimal number 87. Dozen, gross, and great-gross are names given to 12^1 , 12^2 , and 12^3 , corresponding to ten, hundred, and thousand in decimal notation. Most of the modern electronic computers which manipulate numerical quantities use the binary system, with base two, as the fundamental method of storing and manipulating internally. A binary base uses only two digits, zero and one; therefore more places (slightly over three times as many) are required to represent a number than in the decimal system. For example, when the decimal number 55 is to be expressed in binary notation, we break it up into powers of two ($32 + 16 + 4 + 2 + 1$) and it becomes

$$1 \times 2^5 + 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0.$$

Since position now corresponds to the appropriate power of two, the decimal numeral 55 becomes the binary numeral 110111.

For convenience in checking internal operations of digital computers, desk calculators like those in commercial use have been constructed using the base eight.

Development of Hindu-Arabic Number System

Arabic numerals, which are used almost universally today, actually came to us from the Hindus. Not only was a decimal positional system with base ten used in India, but more than a thousand years ago several of the symbols representing digits had attained the form we presently recognize. This notation included a small circle to represent zero. A Hindu mathematics book using these symbols was translated into Arabic, and then into Latin.

In this way the Western world "discovered," or was introduced to, a notation still in use today.

It is significant that this notation was distinct from the alphabets used for written language. Numbers had attained a method of construction which was consistent, useful, and distinctive; written languages were expanded to include these new symbols. The form of digits used as building blocks varied considerably until after the introduction of printing in the fifteenth century. Since then only slight changes have taken place, primarily changes for ease or variety in reading.

SIGNIFICANCE OF ARITHMETIC IN VARIOUS CULTURES

Babylon and Egypt

Some of the earliest written material now in existence consists of clay tablets found in Mesopotamia describing certain arithmetic calculations associated with business transactions. The ancient Sumerians and Babylonians very early developed an extremely useful and rather sophisticated arithmetic system. Simple fractions were reduced to fractions with common denominators of 60. Positional notation permitted easy manipulation of large quantities. The significance to business and trade is obvious. Babylonian astronomy also benefited from an arithmetic system that was simple and widely understood. Astronomical observations, whether applied to eclipses, planetary movements, or the calendar could be consistently carried out.

Although some of our oldest mathematical traditions come from the Egyptians, their aptitude in arithmetic was not as pronounced as that of the Babylonians. Neither positional notation nor simple fractional calculations were used by them. Fractions were added only if their numerators were unity, and involved manipulations were required to carry out such reductions. Perhaps it was the awkwardness of Egyptian techniques that tended to restrict their application to a small educated class, primarily composed of the priests. Thus arithmetic and religion became closely associated in Egypt, whereas in Babylon arithmetic had become the tool of commerce.

Greece

Notational problems in Greek arithmetic may have played a role in turning the emphasis of Greek mathematics away from numerical concepts. Routine calculations were considered a task only slaves should perform. Certain aspects of number theory were worthy of consideration by scholars, but only because of the philosophical or logical beauties inherent in them. An illustration from number theory is a *perfect* number. A perfect number is equal to the sum of all its factors except itself; for example, $6 = 1 \times 2 \times 3 = 1 + 2 + 3$. Geometry, with its application of reason to abstract problems, completely overshadowed arithmetic in the culture of Greece. Even the interesting concepts of irrational numbers carried primarily a geometrical significance.

Middle Ages

Arithmetic during the Middle Ages labored under the cumbersome notation of the Romans and the geometrical tradition of the Greeks. With the breakdown of trade in Europe, it is not surprising that arithmetic played little part in the lives of the people. Real progress is difficult if incentives are not present. Only in the Near East and across Northern Africa was there a vital civilization in which business and trade flourished—and in which arithmetic began again to be significant.

The Middle or Dark Ages show a remarkable lack of arithmetic influence on their culture. The monastic tradition, in the midst of confusion and chaos elsewhere, made theology the unifying principle. Neither arithmetic nor mathematics, nor even the entire range of science, was sufficiently powerful to make a real impact.

Age of Discovery

The Age of Discovery has only approximate boundaries. As their imaginations began to stir, men questioned what lay beyond the nearby mountain or across the sea, and numerical ideas again became significant. The ordinary man was interested in reports of voyages and expeditions, and he again considered the basic concepts of science. Since the purely philosophical approach of the

Greeks seemed inadequate to the experimental attack of the explorer, science sprang from its centuries of sleep as a quantitative study of the universe. The rise of commerce made the numerical manipulations of trade more significant than they ever had been before.

Explorations beyond the nearest horizon not only broadened the perspective of the entire populace, but also forced applications of mathematics to physical situations in new and exciting ways. How far has the ship or the caravan gone? How can the required ships or vehicles be built? How can provisions be estimated, or the results of trade evaluated? The age of discovery called for answers to these questions and such answers required an arithmetical adeptness never before needed.

Questions regarding the whole universe found some answers in numerical descriptions as well as in geometrical formulations. Observations of the heavens were carried out consistently and results were sufficiently well recorded so that later generations were able to develop physical principles from them. The Age of Discovery also led to discovery of the experimental approach, and such an approach is closely related to arithmetical representations.

The Industrial Culture

The accomplishments of society still were limited by the amount of energy available from men or animals. Some tasks beyond the capability of an individual could be performed by groups of men or of men and animals. As energy demands increased, however, merely adding energy units of the same size became impractical. Drastic power increases through the harnessing of water and steam followed and led to the Industrial Revolution. The mill, the factory, the mine, and eventually the steamship and railroad became dominant, and the small home industry began to fade.

The industrial culture may not have directly advanced basic ideas of arithmetic, but it did furnish an unusual and critical climate in which arithmetic could flourish. The associated business activities which grew up in order to control industries soon found themselves in a maze of arithmetical records. The industrial culture was controlled by the counting house. Industry, business,

and civilization itself became wholly dependent on numerical records.

ARITHMETIC IN DAILY LIFE

Business and Finance

Much of the impetus given to arithmetic in ancient times came from problems associated with trade. It is difficult to conceive of the development of our modern complex business structure without giving credit to the role which arithmetic played. The number of arithmetic operations associated with even the simplest business transaction is almost staggering.

Consider, for example, the use of arithmetic in the operation of a small department store. There is always a fundamental question regarding stock inventory. No simpler arithmetic problem can be stated; it is merely a matter of counting. And yet the requirements of competitive operation complicate even the counting process. A certain amount of stock may be on open shelves; more is likely to be in storerooms relatively near the open shelves; additional quantities are probably available in warehouses somewhat separated from the main store; and finally goods may be in transit from the manufacturer to the store, or may be committed but not yet removed from the shelves or warehouse. Therefore the results of a whole sequence of counting operations must be combined, and a serious problem of simultaneous measurement must be resolved. The cost of the items involved usually influences the degree of accuracy required.

The simple problem of inventory has recently become involved in attempts to reduce the costs of doing business, as one of the important problems associated with *operations analysis*. This technique attempts to apply frequently complex mathematical methods to the solution of such problems as these: Where should warehouses be located so that orders can be filled at minimum transportation costs? How large an inventory should be maintained for immediate delivery? In the latter case, costs of stocking enough for all demands must be balanced against the loss of good will and business in case the supply is exhausted.

The simple fundamentals of counting—addition and subtrac-

tion—dominate such problems. When one adds the operations associated with billing, payrolls, profit and loss statements—not only for the benefit of the owner, but also for governmental and tax purposes—and other operations required in carrying out a business enterprise, it is easy to understand that the wheels of business turn under the power of arithmetic.

Application of arithmetic calculations to problems of banking and finance are not difficult to imagine, but the extent to which they dominate this field is much more difficult to appreciate. The financial statement of any business becomes its major control document. Manipulations of an accountant represent a direct application of arithmetic processes. Control of large sums of money through numerical means is, of course, not new. Complex as the financial operations of the Medicis or the Rothschilds may have been, however, they are today dwarfed by operations of even the small bank on the corner. Expansion of business led to development of financial tables, business machines, and highly complex communication equipment for transmitting numerical information from one part of the world to another. A listing of the arithmetic operations involved in the sale of one share of stock is long and boring. The amount of planning which goes into the arithmetic handling of a single check is difficult even to describe.

In attempting to reduce the problems of insurance from chance to mathematical quantities, mathematicians developed the fields of statistics and probability. By their nature, these techniques depend directly on the simplest arithmetical processes, and they are an excellent illustration of the way one moves from arithmetic into the more complicated aspects of mathematics. Here we attempt to place numerical evaluations on situations which are never precisely measured. The success which has been attained is all too often taken for granted. The ability to abstract significant numerical information, to analyze its meaning, and to use it in predicting future events is one of the most exciting aspects of arithmetic.

Industry

To the extent that an industry is a part of our business and financial complex, it will share all of the applications of arithmetic discussed above. It is important that the results of a factory

operation—output, costs, orders, and the like, be immediately available to management so that the required decisions may be made.

Much of modern industry requires science, mathematics, and arithmetic for more fundamental purposes, however. Mass production and use of interchangeable parts have led to requirements for accuracy that were unheard of a few years ago. The .01 inch requirements of a generation ago were replaced by .001 and even .0001 in World War II; but precision to ten-thousandths of an inch is now being replaced by accuracy to millionths of an inch in industry today. Parts must be measured or evaluated very carefully even though it means increased costs. This is possible only through statistical techniques by which the examination of a sample enables us to deduce the quality of the entire production run. Under the name *quality control* a complex statistical analysis is reduced to simple charts, rules, and numerical checks which can be used by any inspector or foreman.

The operation of a modern factory with its countless machines and processes must be studied from a mathematical point of view in order to obtain satisfactory efficiency. Are more efficient machines worth the cost of replacing older but adequate units? In what order shall operations be carried out to minimize costs? How shall work be scheduled in order to produce the largest output with minimum manpower, fewest raw materials, or smallest cost? The modern factory is a complex unit which must be operated with foresight and care, and these questions may no longer be answered by intuition; specific numerical evaluations must be made. Many such analyses are described by the term *programming*; that is, the planning of sequences of operations to produce a given result at minimum cost or effort. On a much lesser scale, calculations involved in solving an arithmetic problem can be planned in advance or programmed. Such programming becomes the major task in the use of modern computers.

Government

Some of the earliest recorded applications of arithmetic for governmental purposes are associated with problems of the census. Even before collecting taxes became a prime task of governments,

the census was necessary for determining potential military strength. With population increases, arithmetic problems associated with census material became staggering. The impetus given by census requirements was largely responsible for introduction of machines which use punched cards as a means of processing and analyzing data. Governmental analyses and reports based on census information must be available in a sufficiently short time to influence business and industry, or the whole point of obtaining such information is blunted.

In monetary control, health, general business statistics, and countless other areas, governmental units have become the most important producers of arithmetical information. Governments of all sizes find that arithmetic information on their processes is essential, not only in maintaining control of operations, but also in explaining their significance to the people. Perhaps the growth in our national debt over the past few decades is an appropriate example of the increasing need of arithmetic in governmental processes.

The Home

As recently as a century ago the average home experienced little contact with numerical procedures or arithmetic manipulations. The United States was still a rural nation; few families were dependent on weekly wages; goods which were grown or produced were used within the family group or were bartered for necessities. Today numerical problems exist in every home.

Taxes, of course, furnish an obvious illustration in almost every family. The protests which arose when federal income taxes were first suggested frequently had as their theme the impossibility of an individual's maintaining appropriate records to permit correct payment of taxes. Today such record keeping is a commonplace, not for the few, but for every home in the U. S. If we add all the other taxes which are assessed, we find that arithmetic plays a very active role in any home.

The growth of installment buying, both for capital goods and for consumer items, poses a significant challenge to a teacher of arithmetic. In this area a comprehension of the basic operation is absolutely essential, but it is very rarely obtained. An under-

standing of installment buying and of the hidden charges involved requires utilization of arithmetical principles.

Insurance is important in modern society and the home. The mathematics of insurance, called actuarial mathematics, is very complicated, involving principles of probability and statistics. The basic arithmetic principles which underlie insurance can, however, be understood by the layman. The stake of an average family in insurance—fire, life, automobile, liability, health, and countless other types—is now so great that arithmetic is essential for an understanding of this aspect of our daily life.

ARITHMETIC IN SCIENCE AND TECHNOLOGY

Historians frequently characterize an era in terms of its dominating drive. Such a drive influences all the activities of a country or group of nations, and determines the culture of the period. Imperial Rome might be characterized as a culture of law, the Middle Ages as a culture of theology, and the Renaissance as a culture of awakening. A major era can, of course, be broken into smaller units and cultural eddies identified.

The Culture of Science

For almost 400 years following the discovery of America, this country was dominated by the frontier spirit; there was always the unexploited opportunity to the west. The discovery of a few pebbles of gold in California and in the Yukon created excitement not only among prospectors and miners, but also in the offices, stores, and factories of great cities and on farms that had been tilled for a hundred years. The cultural eddy from the Industrial Revolution swept across from Europe and mingled with the culture of the frontier, combining to create a vital America at the beginning of the twentieth century.

From seeds that were apparent at the beginning of this century, there has sprung up in our country, and around the world, a significant new culture. The understanding of electricity, though not complete, has been sufficient to exploit its powers in countless ways. Fossil coals and oils not only have been applied as fuels for comfort and transportation, but also have been torn apart,

rearranged, and reassembled to give us innumerable chemical compounds for fantastic new applications. We have probed within the hard-shelled secrets of the atom to obtain unbelievable power for constructive uses and for destruction. The threshold to outer space has been crossed and the limitless expanses of the universe are opening to us.

This Culture of Science which characterizes the present era is dominant and demanding. Escape from it is almost impossible; ignoring it is unrealistic and frustrating. It permeates our recreation and hobbies (hi-fi and photography), our entertainment (TV) our food and clothing (frozen foods and "miracle" fabrics), our health (x-rays), and the safety of our nation (DEW line and guided missiles). It must be understood, appreciated, and enjoyed.

Resurgence of Arithmetic in Modern Science

Gauss named arithmetic the Queen of Mathematics, but her crown has frequently been tarnished, or at least obscured, by more striking coronets around her. Greece was so preoccupied with the glamor of geometrical reasoning that arithmetic failed to make real progress except in very limited areas. The Dark Ages were burdened by an overly complex notation, and numerical manipulation suffered. In more recent centuries analysis has often overshadowed the plodding arithmetic, even though analysis owes its vitality to this Queen.

The beginning of the nineteenth century found the career of Carl Friedrich Gauss well begun; this "Prince of Mathematicians" placed arithmetic in its rightful role as a challenging and independent mathematical discipline. He was able to apply his abilities to astronomical calculations and, more significantly, he clarified the relations between algebra and arithmetic.

During the latter part of that century the bounds of arithmetic were stretched by Georg Cantor and others to include infinite numbers. There are always more points on a line segment than correspond to any ordinary cardinal number, and Cantor's infinities were called in to fill this gap.

If the contributions of Gauss and Cantor are primarily significant in abstract phases of arithmetic, a relative unknown, Charles Babbage, had a marked effect on practical calculations. During

the early part of the nineteenth century, he outlined plans for calculating *engines*, which were very similar to modern digital computers. He proposed the introduction of data and instructions into these *engines* by means of punched cards, a variation of those used by the textile industry of his day for weaving designs in fabric. The data would be operated on in an arithmetic *mill*, which would perform simple arithmetic operations in accordance with instructions given. Intermediate or final results would be placed in a *store*. This concept was a major step in reducing mathematical calculations to a program of simple operations which could be accomplished automatically. It is not surprising that construction of such a device was far in advance of mechanical skills available in Babbage's time. Only small portions of his machines were built, but modern computers applying the principles of electronics have been able to achieve the goals he outlined.

World War II forced the exploitation of electronic techniques for arithmetic manipulations. Explosive developments in digital computers since that time have freed modern science from the drudgery of arithmetic routines. The result, based entirely on smashing this arithmetic barrier, has been an opening up of new problems and completely new areas to mathematical analysis.

An increase of information about secrets of science and technology has permitted more accurate and complete mathematical analyses. Such analyses frequently cost less than actual construction and lead to an understanding which cannot be attained in any other way. In many cases a system being constructed is so complex that significant preliminary tests cannot be made; analysis may, therefore, provide the only justification for such a project before full scale utilization. Coupled with the increase in analytical work has been an associated increase in arithmetical calculations. As the burden of such calculations grew, it became necessary to limit conditions being considered to those actually anticipated. Computing machines, however, have permitted a *shotgun* approach to complex problems—a numerical analysis not only of the anticipated situation, but also of variations from it.

Another interesting result of breaking the calculation barrier is given the intriguing name *Monte Carlo*. The usual approach to

problems of chance has been to analyze a game mathematically, then apply the results. In the Monte Carlo method the computer is set up to play a game of chance, essentially rolling dice or tossing coins. The accumulated results of a long series of such games are studied to obtain the solution of the basic problem. And, fortunately, many serious problems can be described in terms of such games of chance—nuclear fission, temperatures in walls of rocket engines, mixing of chemicals, and many others.

Contribution to Defense of Western Civilization

The onset of World War II found a mass of scientific technology available for use in military problems. This triggered an unusual emphasis on the mathematical and arithmetical in our defense program. Since the end of the war this trend has continued and at times even accelerated as the developments, once of interest only in combat, were applied to other situations. Archimedes and Leonardo da Vinci would both find the phenomenon familiar.

The problem of flutter, a destructive vibration of an airplane wing due to high velocity, merited little or no consideration in 1937. Twenty years later, thousands of pages of calculations for a single wing design are necessary to insure safety. Problems which could not even be considered a decade ago, and required two months of calculation on the computers of five years ago, now are carried out in five minutes. During the launching of an ICBM or satellite vehicle, its position is carefully monitored, calculations are made by large computers on the ground regarding its actual flight path, and, almost instantaneously, corrections are determined and transmitted during those few moments of powered flight. Thousands and even hundreds of thousands of arithmetical operations must be accomplished every second. Small digital computers have been developed for use in aircraft to determine the course required for interception of a target, the instant at which to fire guns, rockets, or missiles, and even to establish the landing path and take over for the pilot as he returns from his mission.

The introduction of mathematical operations into the core of national defense is partly due to a desire to remove the human being from areas of known danger. Western civilization places so

high a value on the life of an individual that it is feasible to create a highly complex and expensive system to give that individual a better chance of survival. Push-button warfare attempts to buy lives with scientific complexity; numerical operations are an essential part of such complexity.

Since no enemy is static in his own developments, there is a competitive feature in modern warfare. Combat planes of World War II flew at speeds ranging between 250 and 400 miles per hour; our modern interceptors move at speeds three times as great. Since human beings are not able to carry out many actions as rapidly as required, calculators have been introduced to simplify the work or perhaps to make it entirely automatic.

As an example, the SAGE system for continental defense obtains information from many sources (primarily radar), processes this information through arithmetic operations carried out by one of the largest computers so far constructed, and presents the results in a form by which a battle plan can be executed. Here a highly complex problem which is too large for the comprehension of a single individual must be simplified, preliminary decisions made, and multitudinous activities coordinated with unusual rapidity, accuracy, and completeness. This indicates that as problems of defense become more complex, arithmetic processes become more essential and the necessity for rapid and accurate manipulation becomes paramount.

ARITHMETIC IN THE DEVELOPING CULTURE OF TOMORROW

Impact of Automation

The actual magnitude of the automation problem is a matter of opinion and argument. That the problem exists now and will be of increasing significance in the culture of tomorrow is an admitted fact. Demands for increased precision, rapidly rising labor costs, and manpower shortages during most of the last decade have given impetus to the automation trend.

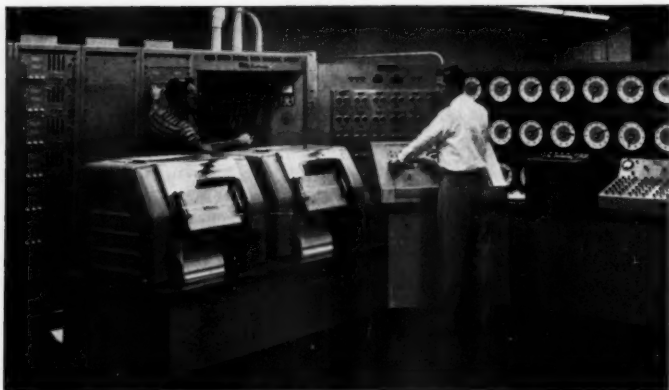
There is a difference between the process of industrialization, or mass production, and that of automation. If it is possible to position a drill in such a way that a hole in a block of material is located properly, very little sophistication is needed to add a

second drill and simultaneously drill two holes. If one machine requires only a small percentage of an operator's time, a second machine can frequently be located alongside it, and the same operator can handle both. The role of the workman in this case is to determine when the operation should be started and when stopped, and to monitor any malfunctions during the processing. He interprets the results at any stage in terms of the goals and modifies the operation in order to attain these goals. The operator is the *feedback* element.

Automation usually attempts to replace this feedback by a process within the machine and uses a human being only to monitor malfunctions of the complete system. A machine which can compare a dimension with a standard and continue operation until the difference between the two is sufficiently small possesses the inherent characteristics of automation. If it is necessary to extract the work, compare it visually with a standard by means, perhaps, of micrometers, and then replace the piece in a machine to complete the job—this is characteristic of little or no automation. The truly automatic process provides for decision-making internal to the process itself.

A distinction can be made between the use of analog or similarity operations and digital or numerical operations in automation. The analog device depends on principles of similarity, which may be geometrical, electrical, or mechanical, or may take an entirely different form. A wind tunnel is an analog of full-scale airplane flight; an electrical circuit may be the analog of a mechanical device; a chemical process may have a complex electro-mechanical analog. If the automated system compares its results with the goal by means of an analog device, this may be called analog automation. Many examples of automation are of this type. Their success has been outstanding, particularly in the fields of process control. A \$10 million chemical plant completely controlled from a large panel by two men is in operation. Milling machines that can precisely duplicate a complicated model have been in existence for some time.

A more significant recent development is that of a numerically controlled machine. If a complicated object is to be cut from a piece of metal, the dimensions of the final object, the equations



of the curves or surfaces involved, and the location of specific significant points are fed into a digital computer. This computer, and the mathematicians associated with it, create from the data a numerical program, or sequence of operations, on punched cards or on magnetic or paper tape. When the machine has accomplished these operations, the completed object will be precisely the one intended by the designer. Just as mathematical problems are planned in detail, or programmed, for a computer, so can the work of a machine be precisely programmed. Machines of this type for working metals are important in our production facilities today.

Application can, of course, be made to far more complicated processes than the programming of a single machine by arithmetic means. As digital computers become larger and more complex, it becomes easier to simulate in the laboratory a complete factory, or at least the whole of a complicated process, work out the bugs, feed instructions to a factory, compare results with requirements, and carry out a completely automatic manufacturing operation.

The speed with which digital computers are able to operate is a limiting condition on their application. If the real process is to continue without interruption while calculations are made (that is, if the computer is to operate in real time), the calculations must be simple or the computer must be capable of high speeds. Modern arithmetic units are frequently capable of 15,000 calcula-

tions per second and this will be increased to 100,000 per second on machines now being built. Research in solid state and low temperature physics seems to indicate that a million or more calculations per second will soon be seriously considered. Such speeds permit precise calculation as quickly as a human being can apply intuition.

Automation will not invade every field of activity. Economic considerations, rather than the feasibility of applying the techniques of automation, will be limiting factors. But any consideration of the culture of tomorrow must involve an evaluation of the impact of automation.

Impact of New Discoveries

Even though our imaginations have been stirred by discoveries of the first half of this century, it is difficult to extrapolate very far into the future. The nature of scientific progress has always been hard to predict. Most pure research which has been carefully carried out has fitted into some significant work, unexpected though that work might be. It is logical to expect that almost every research activity now being carried out will affect the culture of tomorrow.

It is not difficult to imagine the place of numerical processes in scientific and technological discoveries. As the accumulation of knowledge increases, the discovery process becomes more complex, emphasizing still further the importance of mathematics and arithmetic.

It is, perhaps, less obvious that new discoveries in the fields of human relations will have arithmetical overtones. Many studies regarding the workings of society have their foundations in statistical analyses. Economics, perhaps because it so frequently deals with monetary values, was one of the first of the social sciences to utilize arithmetic approaches. Psychology, sociology, politics, and many other areas involving the individual and his society may be expected to show pronounced development, partly based on arithmetical analyses.

The time has disappeared, probably forever, when a single individual could understand the whole field of science or human affairs. Too much knowledge has accumulated to allow the

existence of a "universal genius." Da Vinci could paint "The Last Supper," design architectural masterpieces, develop aerodynamic concepts (the helicopter), act as a military strategist, and, in general, know something about almost everything. In more recent times Benjamin Franklin showed a similar range of capabilities. Because of the extreme specialization of today, only a small percentage of the mathematicians at a professional meeting are able to understand all the mathematics papers presented. This specialization influences the whole field of discovery. It is necessary to have team research in accomplishing modern discovery, because the individual cannot comprehend all the problems associated with a complex situation. The problem of communication is fundamental in the operation of any team. Even when team members speak the same language and work in the same area it is difficult to communicate precisely without using mathematics as the language. If mathematics is the language of precise communication, arithmetic may be considered its alphabet. When development of a discovery requires use of information from another country or from another era, mathematics and arithmetic again become the significant links.

The impact on our lives of recent discoveries and their applications may seem too obvious to require comment. Electricity, radio, TV, the atom, the guided missile, and the satellite are usually taken for granted by youngsters even though their teachers are conscious of the changes introduced by these developments. New discoveries will have a pronounced impact and the student must be prepared to evaluate and understand them. A century ago it took about six months to cross the United States by wagon train; today a man-made satellite crosses it in 8 minutes. Commercial flights are made over the North Pole and the South Pole. Whole nations are moving from the crudest civilization into the modern atomic age without passing through any of the intermediate stages which we found so time-consuming and painful. Mankind stands on the threshold of outer space, and opportunities of exploring the farthest reaches of the universe in person are at hand.

Regardless of the nature of new discoveries or the processes by which they are attained, it is certain that they will have a pronounced impact on the future.

The Individual in a Numerical World

The culture of science, which forms the basis of the developing culture of tomorrow, exists in an increasingly numerical world. What is the role of the individual in such a world? How can he adjust himself to it, avoid being overwhelmed, and make a significant contribution to its existence? This is one of the fundamental educational problems of the present.

It is impossible to educate the individual properly without paying attention to the dominating forces in his environment. The world of today with its scientific culture forms the most logical basis for an understanding of the world of tomorrow. The cultural environment of the individual will be heavily dependent on the numerical approach.

It is not difficult to persuade teachers, parents, and even the children themselves of the necessity of understanding mathematical processes for scientific use. There has been sufficient emphasis on the relationship between number and science so that the arguments need only be stated, not developed. For specialists in science and technology, numerical manipulations must be understood, become routine, and be made as certain as instinct. The student who may be capable of unusual scientific achievement can be completely discouraged from any mathematical interest by poor or uninspired teaching in the early grades. Unfortunately this interest may be difficult or impossible to arouse during later years.

If the culture of civilization is so heavily influenced by science, it is necessary for the non-specialist also to understand the basic building blocks of his civilization and obtain some understanding and appreciation of mathematics. It should be a great challenge to the teacher to "get across" basic numerical principles for the student who will not be a specialist in science. An appreciation of the relations among quantities is essential in a society built on numbers.

Methods of impressing on students the significance of numbers and their manipulation need not be detailed in this chapter. The alert teacher who finds geographical and historical references in the daily newspaper and on the radio can find numerical and arithmetical implications in much of the world around him. The frequency of a musical tone, distance from the earth to a satellite,

dimensions and weights of everyday objects, temperatures and weather phenomena—all these and countless more can help to point out our dependence on numerical processes. Better still, an early tie between the student and the world around him through numerical relations removes the fear of arithmetic and mathematics exhibited by so many adults, a fear which prevents them from appreciating the scientific culture of today and tomorrow.

It must be emphasized that we do not imply, by our attempts to interpret our developing culture as one dominated by numbers, that the future is cold, uninteresting, or without excitement. The physicist who understands the structure of an atom is capable of a rich appreciation of the electric light which runs from an atomic-powered generator miles away. A Bach fugue can be more than interwoven melodic lines when the mathematical structure of the music is also understood. The precision of a fine automobile engine, the sturdy but soaring beauty of a skyscraper, the complexity of modern transportation, the marvels of space conquest—all these furnish examples of a rich and exciting technical society. The individual who is capable of appreciating, even though he does not completely understand, the role of mathematics in the complex technology with which he is surrounded finds an interest in the culture of tomorrow which the mathematical illiterate cannot see.

Writers have often deplored the fact that advancing culture seems to stultify the individual. They threaten that automation will dominate man and that man will be destroyed by his own attempts to defend himself. There is merit in such concern. Only when the ordinary individual begins to grasp the significance of the world around him will he be its ruler and not its slave. Arithmetic is the alphabet by which the culture of tomorrow must be understood.

Structuring Arithmetic

H. VAN ENGEN and E. GLENADINE GIBB

HOW DO PEOPLE LEARN? Under what conditions do they learn most effectively? What, if anything, characterizes easily learned materials? These and similar questions have long interested psychologists, philosophers, and educators. As a result of this interest, various theories have been developed to explain the learning process. Such schools of psychology as Connectionism, Associationism, Field Psychology, Behaviorism, and Functionalism have taken learning as an integral part of their domain of interest, although these schools do not have a common concern for the same aspects of learning.

Early in the twentieth century there arose in Europe, and primarily in Germany, a school of thought which emphasized patterns, forms, configurations and structure as phenomena of learning. This school of thought, known as field psychology, placed an emphasis on such ideas as:

1. A problematic situation is an unstructured situation. Insecurity accompanies unstructuredness. When the problem is

structured we feel secure; we have solved the problem; we have learned.

2. Repetition may change the way we have structured a situation. However, the better the structure, the less the change brought about by repetition and the fewer repetitions required.

3. Changes in structure come about, at least in part, according to principles of patterning in perception (that which we see). In other words, the problems of acquiring knowledge are believed to be closely related to the laws governing perception.

4. When we feel secure, having structured a problematic situation, we say that we have developed an insight into the problem. Insight is dependent upon the awareness of relationships between parts and of parts to whole.

5. Learning is structuring, forming patterns, making configurations.

In the United States, Gestalt principles, the most familiar of the field theories, did not make much headway until the third decade of this century. Their influence on arithmetic can be felt, in part, through an emphasis on meaningful and insightful teaching. In recent years the term *structure* has been appearing more frequently in the literature on the teaching of arithmetic than it did in the past. The frequent use of the terms *insight* and *structure* may be a gauge of the extent to which Gestalt theories are becoming acceptable to students of arithmetic.

Catherine Stern is an advocate of structural arithmetic in the United States. A former research assistant to Max Wertheimer, one of the founders of Gestalt psychology, she had ample opportunity to develop a method of instruction which incorporated its principles (12). The reliance on Gestalt (structural) theories of learning in arithmetic (*that which we see*) is illustrated by the following quotation (13:177):

The child is presented with a variety of concrete materials which make the structure of the number system visible to him. As the child uses these materials with his own hands, he discovers all the number facts. . . .

Here we see that what the child does with concrete things is supposed to develop the concept of a number system.

STRUCTURE—WHAT IS IT?

Some of the more significant words in the literature on arithmetic are difficult to define, and some may never be defined. How can we explain *concept*, *meaning*, *insight* and *configuration*? To this group we must add the word *structure*. The best that one can do is to define it by illustration. This we will try to do.

One speaks of the *structure of a building* and the *superstructure of a bridge*, and builders use *structural steel* to provide the main support for walls, floors, and the like. All these are readily recognized. They are, for example, the heavy steel beams, the 2 x 4's and 2 x 6's, that make the skeleton of the building. The various members of this framework fit together to give the building its strength, form, shape, and character.

So it is in the realm of ideas and concepts. Ideas fit together to make a whole—a structure. Why does one recognize "The Star Spangled Banner" whether sung by the deepest bass or by the highest soprano? Because there is a structure of tones—a pattern of sounds—that is the same in each instance. This structure is recognized even though no two notes in the different renditions are identical.

Most of the think-of-a-number puzzles are based on a hidden structure. The puzzle would not be fun if its structure and pattern were evident to the individual attempting to solve it. Consider this puzzle: Select any number; multiply it by 5; add 6 to the product; multiply the sum by 4; add 9 to the product; and multiply the sum by 5. If you have performed the necessary calculations, you should have a number of at least three digits. Cancel the last two digits; subtract one from this result and you arrive at the number initially selected.

Puzzling? Certainly! The elements of the puzzle are given in such a way that the structure is not evident. All the mystery vanishes when one sees how the various operations fit; that is, when the situation is structured the puzzle is no longer a puzzle. To see this, we will exhibit the structure.

Let N hold a place for the original number selected. Then $[(5N + 6) 4 + 9] 5 = 100N + 165$. The last two digits of $100N +$

165 are 6 and 5. When these are cancelled, the number $N + 1$ remains. Subtract one, and the result is the original number.

The end result of structuring the puzzle is insight—an important concept for arithmetic teachers.

Consider now an example drawn from the arithmetic classroom. One third-grade child knew the number words—could count—from one to one hundred. However, when asked to illustrate what she thought 26 meant she placed 26 single counters on the table. She did this in spite of the fact that she had handled the sticks and knew that there were bundles of ten sticks on the table. When the teacher asked her if she could have used the bundles, she said “Yes” and proceeded to take the bundles apart and lay out the sticks, counting them one by one.

This child knew the word patterns or verbal structure of our numeration system, but the essential idea, the structure of the decimal system, had escaped her. She did not know that we essentially count “... nine, ten, one-ten and one, one-ten and two ... two tens, two tens and one ... five tens, five tens and one, five tens and two ...” This child, having had no structured visual experiences with tens-blocks, bundled sticks, and the like, had failed to conceptualize the counting process. She had a poor foundation on which to build her future work in arithmetic.

An example of *seeing* structure is supplied by this real-life experience with Ann, a child who was less than successful with fourth-grade arithmetic. The description which follows involves work with the *teen* numbers.

The material used to study the teens included: (a) sticks and cards in bundles of tens and ones, and the spool board, (b) dimes and pennies, and (c) an abacus. A sample of the work done can best describe the procedure used to get Ann to see the structure of the teen numbers.

Fifteen are ten and five. Move two from 10-group to 5-group and we have eight and seven; one from 10-group to 5-group and we have nine and six. Later we combine groups. Six and seven, for example, were first *seen* and then *thought*, as added by moving three from the 6-group to make a ten of the 7-group; three and ten, thirteen.

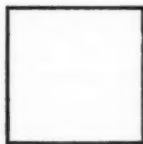
Ann needed to manipulate teens-groups before she could think

the recombinations with the help of coins or abacus. One week later she no longer needed groups, but she thought slowly. Two weeks later, a total of five weeks, she no longer had to think. The answers, in her words, "just pop into my head." When answers began to "pop into" Ann's head, we began to use figures and signs to express the putting together and taking apart of groups. *But the idea was first.* Once Ann sensed the structure of the teens combinations situations, she was ready to begin symbolizing the idea by such symbols as $6 + 7 = 13$.

Space limitations prohibit our giving further illustrations of the structuring of ideas. However the reader can supply others from his own experience. Everyone has had experiences in which the various elements have suddenly fallen into place to form a pattern. This is the "Oh! I see" moment. We say that insight has occurred or that one has structured the situation at that time.

TWO KINDS OF STRUCTURE IN ARITHMETIC INSTRUCTION

Arithmetic has long been taught without asking certain basic questions. How or where do children acquire numerical and geometric ideas? Most certainly they are not innate. If this is granted, then it becomes evident that the child must acquire mathematical ideas from events within the physical world. He recognizes similarities—structural similarities—and learns to associate names with the structure. For example, he learns that the figure shown below



is a square and that all figures so structured are called squares, regardless of size. To do this, the child must recognize similarities (likeness of parts) and ignore differences (lengths of sides). He sees that the structure consists of four equal sides and four right angles and he fits these together to make a whole. All this takes

place even though there may be no vocabulary to express the ideas.

Physical or concrete experiences which are patterned lead to the development of concepts; these concepts are themselves a pattern of physical elements which are similar to the concrete experiences. We are thus led into two realms of structure; the first, a physical structure (the sequence of events and arrangements of objects); the second, a psychical structure (that which causes a given situation to make sense so that everything fits).

Consider how the young child forms the concept of *chair*. He encounters various pieces of furniture in the home and soon learns that one particular piece of household furniture is called a chair. He cannot be induced to call a table a chair even though there are physical similarities between a chair and a table just as there are structural similarities between the concept of chair and the concept of table. Though there are vast similarities, the child sees that one class of objects (chairs) fits one pattern and the other class of objects (tables) fits another pattern. These are the physical similarities and differences from which concepts arise. The concept of chair has in some way much the same characteristics as the chair itself. All of this suggests that concepts have a structure which, according to some psychological theories, are similar to the physical characteristics of the objects.

The thought that our ideas and their structure come, in their inception, from the physical world itself is nicely illustrated by Wertheimer in his book *Productive Thinking* (15:40-41). This analysis of how a child learns to find the area of a parallelogram is insightful and we can do no better than to quote in detail.

1) The problem is confronted: what is the area of the rectangle? I do not know. How can I get at it?

2) I feel there must be an *inner relation* between these two: the size of the area—the form of the rectangle. What is it? How can I get at it?

3) The area may be viewed as the sum of the small squares in the figure [footnote omitted]. And the form? This is not *any* figure, not any heap of small squares in any form; I have to understand how the area is built up in this figure!

4) Are the little squares not *organized* in this figure, or *organizable* in a way that leads to a clear structural view of the total? Oh, yes. The figure is throughout equal in length, this has to do with the way the

area is built up! [Figure omitted showing rectangle made up of small squares.] The parallel straight rows of small squares fit together vertically in mutual equality, thus closing the figure: I have rows equal in length throughout, which together form the complete figure.

5) I want the total; *how many rows* are there? I realize that the answer is indicated by the altitude, the side a . How long is *one row*? Obviously this is given by the length of the base, b .

6) So I have to multiply b by a ! (This is not multiplication of two items equal in rank: their characteristic functional difference is basic for the step.)

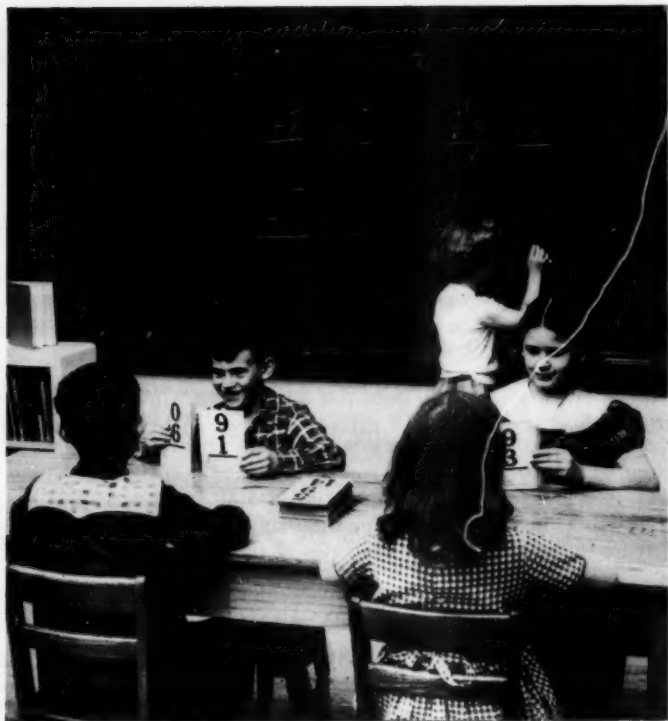
In this structurization of the rectangle, the question of the area becomes clear. The structure obtained is seen comprehensively and transparently. The solution is reached [footnote omitted] in the realization of the inner structural relation between area and form.

Wertheimer goes on to describe the process whereby the structure of ideas is abstracted from experiences with elements of the physical world (15:41):

There is *grouping, reorganization, structurization*, operations of dividing into sub-wholes and still seeing these sub-wholes together, with clear reference to the whole figure and in view of the specific problem at issue.

In addition to noting Wertheimer's emphasis on the structuring of the idea of area, we should see how closely his method relates the physical experience to the mathematical idea. Here we see a physical experience closely fitted to a mathematical idea. Teachers of arithmetic would do well to study this example closely and then evaluate visual aids for closeness of fit to the mathematical ideas they are striving to get the child to structure. Many perceptual experiences (visual aids) are totally unsuited to the purpose for which they were built because they do not closely fit the mathematical idea. Johnson attested to the importance of close fit when he wrote (7:239), "Conceptualization is therefore easier the more it resembles the perception of concrete objects."

We have seen how ideas are abstracted from the physical world, and we have seen how teachers can help children abstract these ideas out of the physical world by arranging perceptual experiences which closely fit mathematical ideas. Now that we have traced a possible source of mathematical ideas, we are confronted with the very real problem of inquiring whether these ideas (area,



fraction, ratio, addition, and square root, among others) are merely accumulated in some kind of intellectual basket or whether they are themselves structured so as to form patterns.

Of course the answer is that they are made to fit a scheme; they are structured. In making this structure, mathematicians have found that certain abstract elements are the basic units on which to build. Such elements include the *commutative law of addition*, ($2 + 3 = 3 + 2$); the *associative law of addition*, ($(5 + 1) + 6 = 5 + (1 + 6)$); and the *distributive law*, $2(3 + 4) = (2 \times 3) + (2 \times 4)$. There are others, of course, but not an endless array.

The use and misuse of these ideas are illustrated by the child's wanting to subtract 6 from 4. That is, $4 - 6$. The child is told,

"We can't do this." A better answer would be, "With the numbers we have in arithmetic, we can't do this." Here we encounter what the mathematicians call closure, meaning that there is no natural number associated with the symbol $4 - 6$.

Judd, of whom it could be said that he was born 30 years too soon, must have had the two kinds of structure in mind when he wrote concerning "The Arithmetical and Situation Phases of Problems" (8:80):

The results secured by analyzing the textbooks can be clearly stated only after certain distinctions are drawn. The first of these distinctions has to do with the difference between what may be called the "arithmetical" and the "situation" phases of a problem. When a problem deals with pencils or with articles of food, the concrete situation which must be more or less clearly in the minds of the pupils is very different from the concrete situation to which reference is made in a problem dealing with the hours shown by the clock or the distances covered in a journey. These are the phases which are referred to as "situation" phases.

When situations are described in a textbook in arithmetic, they are always rendered more or less precise by the use of quantitative or numerical terms, which are, strictly speaking, not descriptive phases of the situation but facts of the purely arithmetic type. This part of the problem uses Arabic numerals or number names and also such an interrogative expression as "how many" or "how much." The numbers and the expression "how many" or "how much" are arithmetical phases of the problems and are to be distinguished from the description of the concrete aspects of the situation.

Any given arithmetical phase can appear in connection with a vast variety of situation phases, and any given situation phase may, under different conditions, appear in combination with various widely different numerical expressions.

While Judd does not make a clear distinction between the structure of perceptual experiences and the structure of mathematical concepts, one can readily see that this is at the heart of his distinction between the "arithmetical" phases and the "situation" phases of arithmetic. The authors recognize that Judd was not a Gestaltist, although his term *generalization* must have embodied much of what the Gestaltists mean by *structure*. Had arithmetic teachers followed and amplified Judd's analysis more closely, generations of children might have been saved the task of memorizing a patternless arithmetic.

FITTING THE EXPERIENCE TO THE IDEA: ILLUSTRATIONS

To sharpen the idea of getting a close fit between the structure of an experience and the structure of a mathematical idea, consider some very elementary illustrations of utmost importance for mathematical instruction. We will describe the structure of these two situations in adult terms rather than in the language in which children meet such situations in the elementary school. The illustrations will be better understood if the reader will actually take objects and carry out the actions as indicated.

In arithmetic the child is confronted with many situations of the following nature. By way of verbal problems he is told that here is a 5-group (5 apples, 5 pennies, 5 cookies). Then something is said to indicate that a 3-group joins the 5-group. The question now becomes, "What is the number associated with the resulting group?"

Let us list the elements in this experience, that is, the things the child sees. First, his attention is called to a group of 5 objects. Second, he sees 3 objects join the 5 objects. Third, he sees a larger group of 8 objects. And fourth, he is told that this is addition and that the sequence of events is symbolized by $5 + 3 = 8$. Other experiences, similarly structured, lead the child to the generalized notion that addition is associated with the joining of groups. We will call this Experience A.

Now consider the *how many more are needed* problem in arithmetic. Note that its structure is very similar to the first experience (that of joining a 3-group to a 5-group). Again, by various combinations of words, the child is told that here is a 5-group. Another group joins the 5-group but we are not told how many objects it contains. Now the child is told that there results an 8-group. He is asked to determine the number associated with the joining group. We will call this Experience B.

Let us list the elements in Experience B as a child might see it.

First, there is the 5-group just as in Experience A.

Second, just as in Experience A, a group joins the 5-group. (Here a clever switch is pulled; the child is not told the size of this group.)

Third, the inevitable final group appears, as in Experience A,

but again we vary the situation by telling the child the number of objects in this group.

Fourth, we ask the question, "How many objects are in the group that joined the 5-group?"

If the reader has been acting these situations out in the sequence of their occurrence, he will see that physically they are similarly structured; in fact the structural similarities vastly outweigh the dissimilarities. The tragedy of arithmetic instruction is that we fail to take advantage of these similarities; we throw them to the winds by telling the child that Experience A is an addition problem and Experience B is a subtraction problem. Is it any wonder that arithmetic is hard to learn? Children recognize these visual structural similarities and expect to have this similarity carry over into the world of symbols. One child said, "Oh! It's just like an addition problem, but you subtract." The physical structures of Experience A and Experience B require that we symbolize them by combination of symbols that bear a similarity. Experience A should be symbolized by $5 + 3 = N$ and Experience B, by $5 + N = 8$.

The point can be strengthened if the structure of an actual subtractive experience is exhibited. We will call this Experience C.

First, a group of known size (let's say 8) is presented.

Second, by various combinations of words the child is told to remove (or forget about) a subgroup whose associated number is 3.

Third, the child's attention is then centered on the remaining group (*How much is left?*).

It is evident that this experience is not structured as is Experience B. To tell the child that both are symbolized by $8 - 3 = N$ would seem to be very unwise.

Experience A and Experience B are very similar. Each can be symbolized by the general form $a + b = c$. Experience B and Experience C are almost totally dissimilar (physically) and should not be expressed by combinations of symbols that are similar. To do so violates the sense of appropriate symbolism and causes a poor fit between the physical situation and the mathematical symbolism.

That children do not see Experience B and Experience C as the same is attested to in research reported by Gibb (5); that

children do not see Experience B (the *how many more are needed* situation) as a subtraction problem is immediately inferred from an unpublished doctor's thesis by Crumley (3) which is on file at the University of Chicago. That experienced teachers have noticed that children's thinking is influenced by the physical structure of a situation is attested to in a paper by Moser (11). Allport (1:630) recognizes event-structuring and organism-structuring, as is evident from this quotation:

Surely, some kind of event-structuring goes on in the organism, or in the cortex, when an act of perception takes place. This structuring, moreover, is not a human artifact, but a part of the very constitution of the organism, and, as such, is not particularistic, but *highly lawful and general*.

It would seem that it is time to examine carefully the physical situation (visual aids and verbal problems) from which we expect a child to formulate a mathematical structure. In the remaining pages of this chapter we will consider some phases of mathematical structure and their counterparts in the physical world.

USING PHYSICAL STRUCTURES TO BUILD MATHEMATICAL STRUCTURES

The first reaction of the reader to the topic under immediate consideration may be, "What! A structure for a number like six?" Before commenting on this question it may be well to sketch the position of those who wish to *structure* a 6-group, for example.

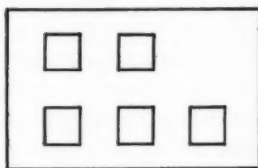
The Numbers From One to Ten

Recent years have seen an ever-increasing demand that the first years of arithmetic be based on the study of groups. Such a program requires the ready recognition of groups associated with numbers less than 10. How is this ready recognition to be achieved?

The usual program requires that the child learn a series of number names largely as nonsense symbols; that is, he must learn to say, "One, two, three, four" before he can count a group of four. This procedure, some say, leads to confusion. A child

tends to count four objects, pushing aside one at a time, and at the count of four holds the last object in his hand, meanwhile looking for approval from his teacher. His actions show clearly that the thinking is fourth (ordinal thinking) and not four (cardinal thinking). He does not understand that *four* is associated with the whole group and not with the object handled last in the sequence. The classroom experience, in this instance, is usually set up in such a way that the two concepts (ordinal and cardinal) cannot be clearly identified for the child. In other words, the experience doesn't fit the cardinal idea. The child does not see the cardinal idea because the situation fails to get him to focus his attention on the group.

Stern (12), among others, feels that the answer to the confused situation described above is to teach group recognition as one of the first skills in arithmetic—even before formal counting. In order to do this, each number representation is structured, that is, a group of five is recognized because the five objects (blocks) will fit into a pattern board with appropriate depressions cut into it. Thus *five* appears as shown below.



A child learns to recognize five objects in this pattern. He also learns to recognize five objects not in this pattern, because he sees that they can be placed in the five-pattern board.

Similar patterns are developed for all numbers up to and including ten. Of course, after sufficient experience with these patterns the child learns to recognize groups out of pattern and, at the same time, he learns to count. (Stern places more emphasis on sticks marked so as to resemble cubes fastened to form a column of five.)

The rise of such patterns (structured groups) to facilitate group recognition is excellent. They enable the teacher to present the cardinal numbers vividly. However, a word of caution should be



introduced. The numbers from one to ten are not actually structured as illustrated. The number six possesses no such structure as $\begin{smallmatrix} xxx \\ xxx \end{smallmatrix}$. *Six* is merely a name given to a property of a class of groups, each group possessing the property that all its members can be put in one-to-one correspondence with all the members of each of the other groups in the class. Also, the child must eventually learn to ignore the pattern idea for these groups. He must deal with the concept of six without thinking of the pattern.

There is still another caution to bear in mind. The patterns under consideration in the last few paragraphs are not important mathematically; in fact, a mathematician totally ignores them. They do not play a part in building mathematical structure. They are nothing more nor less than learning patterns. They make the task of learning to count easier; they supply a sound foundation for subsequent learning, but they are not mathematically important.

The same can be said about many generalizations so frequently encountered in the literature on teaching arithmetic—for example,

the near doubles in addition ($6 + 7$, $7 + 8$, etc.). While it is possible to remember that $6 + 7 = 13$ by recalling that $6 + 6 = 12$, one must look upon these generalizations as a means to help a child learn his number facts. As such they may be a teaching aid, but they will be of little or no use to the child in his subsequent study of arithmetic. Arithmetic is structured according to principles other than those which are frequently used to make certain facts easy to remember.

The Decimal System of Numeration

Once the child has learned to work with groups of one through ten, he should be introduced to the structure of our decimal system of numeration. This structure is important from both a social and a mathematical point of view. Here is one of the child's first steps in learning a structure of mathematical importance.

At this stage the child must learn that the human race does not invent new number names for each group of more than ten. He learns that we count by using new combinations of old names. To teach him this, we again resort to building a physical structure which the child can transform into a mental structure, a concept. This he can do after having had many experiences with groups of tens. We will describe briefly how this may be done.

Let us suppose that for quick recognition the 10-group has been presented in the following form.



We now introduce another object and we have the form



which we call eleven, but we point out to the child that it is really a 10-group and a 1-group. Successive placing of other objects with the 11-group builds up physical patterns and each pattern

is matched with a number word until the child arrives, for example, at the pattern below.



This pattern he recognizes as thirty-four and he sees it as three tens and a four.

Physically structured objects which closely fit the structure of our numeration system can take many forms. It need not be the pyramidal form, obviously. Bundles of ten, blocks glued together to make ten; anything to establish a 10-group for quick recognition. When numbers greater than 99 are to be structured, it would be desirable to have bundles of 100 sticks available. The method for doing this is an obvious generalization of the procedure just described for the numbers from 10 to 99.

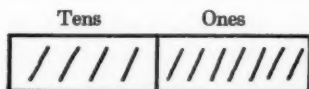
But the task is not complete until a symbolism has been developed for the groupings of objects just outlined and the pattern of this symbolism associated with the patterns discussed in the last few paragraphs. There are many ways to do this. The child can be taught to use a tally system even before he has fully mastered the group formations outlined above. For example, he may be presented with a collection of objects which he must enumerate. He organizes them in bundles of ten, and as he organizes a ten he tallies it in a tally box like the following.

Tens	Ones

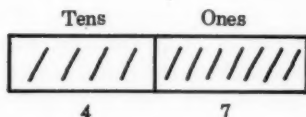
He finally arrives at a stage where no more 10-groups can be made. The tally box may appear as shown below.

Tens	Ones
////	

He next tallies the remaining ones, and there results the following scheme.



This tallying stage is transcended by recording the results immediately under the box.



He learns to read this as forty-seven. This tally box, together with the objects organized as four tens plus seven, supply the perceptual experience from which the child abstracts the concept 47. Similar experiences with numbers between 10 and 99 establish the whole sequence of numbers and the structure of our decimal numeration system. To extend this concept, one would provide experiences with groups of hundreds, tens, and ones in much the same way as outlined for the number 47.

What about generalizing this structure? For example, how can we arrive at the concept of a number like 1,346,921? It is evident that we cannot readily present 1,346,921 objects without considerable labor. Certain things can be done with the thousands but even this is somewhat unhandy. One arrives eventually at the stage when the child is asked to imagine that ten 100,000's are grouped to make a million. Here the pattern of the numbers less than 100,000 should help him, provided our teaching has been such that the pattern has been made evident.

Order

The child is introduced early to the idea that the cardinal numbers (whole numbers) can be ordered. By order we mean that 4 is greater than 3 (written $4 > 3$). This is a basic idea and one that is much used when the whole field of number is structured.

In the fifth grade the child must learn that the rational numbers can be ordered, that is, $\frac{3}{4} > \frac{1}{2}$; in the high school he learns that there are numbers which cannot be ordered.

In the lower grades the child learns that $10 > 9 > 8 > 7 > 6 > 5 > 4 > 3 > 2 > 1$ by observing, for example, what must be done to a 4-pattern to get a 5-pattern. Thus, $::$ is a 4-pattern. If another object is added, there results $:::$; this *put-one-more-on* idea is the physical experience which fits the mathematical idea designated by *one greater than*. In much the same way, the child learns that if another object is added to a 29-pattern, there results a 30-pattern, so $30 > 29$. In time, he forgets about patterns—this must happen—and he merely examines the numerals. Thus, $3,412,115 > 3,412,015$. As we glance along each numeral from left to right, we find the first figure where they differ. This occurs in the third place from the right and so we see that the number represented by the numeral which has the greater digit in this place is the greater number. This is a comparatively mature concept which can be built up with such numbers as 382 and 372 by using both symbols and objects.

We have now developed two mathematical elements by means of physical structures which fit mathematical ideas, namely, the system of numeration and the concept of order (greater than). These ideas eventually become elements which build a mathematical structure.

The Commutative Principles for Addition and Multiplication

Early in the arithmetic program the child learns that a group of four joining a group of eight can be symbolized by writing $8 + 4 = 12$. He also sees that the same result is obtained if a group of eight joins a group of four, namely, $4 + 8 = 12$. Here we see a very nice illustration of a physical situation fitting a mathematical idea. The eight joining four and the four joining with eight fit the idea of the commutative principle for addition: $8 + 4 = 4 + 8$.

In much the same way, the child learns that $4 \times 3 = 3 \times 4$. This can be accomplished by looking at the rows and columns of the following pattern.



Children usually encounter this principle when they meet the product 12×3 . Rather than write $\begin{array}{r} 12 \\ \times 3 \\ \hline \end{array}$ they are told to write

$\begin{array}{r} 3 \\ \times 12 \\ \hline \end{array}$. Previous work with the commutative principle would make this interchange understandable.

Here we encounter a situation in which an important mathematical idea is frequently looked upon merely as a learning device. The child is told to remember that $4 + 3 = 3 + 4$. Why? Because if he remembers this, his work in memorizing the addition combinations is reduced by 50 percent. This certainly justifies its use, but it does not justify neglecting the principle once the basic combinations have been mastered.

To complete the study of the commutative principles, the child should learn that subtraction and division are not interchangeable; $4 - 3 \neq 3 - 4$ and $6 \div 2 \neq 2 \div 6$ (\neq means does not equal).

The Associative Principles for Addition and Multiplication

These principles, while used in arithmetic, are never highlighted for the child in such a way that he is conscious of their use. Possibly this is because they seem so inevitable.

To add $3 + 4 + 6$ one must group because the child learns to add only two numbers and mathematics recognizes addition as an operation on two numbers. The natural procedure is to add 3 and 4 first, and then add 6 to the sum. However, it is possible to add 4 and 6 first and then add 3 to the sum. Each procedure gives the same answer. In mathematical symbols, we write $(3 + 4) + 6 = 3 + (4 + 6)$. This illustrates the associative principle for addition. Children employ this principle when they skip around in an addition column to find easy combinations.

The associative principle for multiplication is much the same. The product $3 \times 2 \times 6$ is found by grouping $(3 \times 2) \times 6$.

However, $3 \times (2 \times 6)$ will produce the same answer. Children can use this principle when finding products $26 \times 2 \times 5$; it is easier to think $26 \times (2 \times 5) = 26 \times 10 = 260$. Proper grouping helps here. There are, however, more important uses; one of these will be mentioned in the next section.

The Distributive Principle

This is the last of the important principles we mention here. These five principles and the numbers introduced in the first few grades of the elementary school make up the number system taught in the elementary school. (There are, of course, many more number systems. For example, the numbers on a clock form a finite number system.) By means of these principles all our computation with whole numbers can be organized into an understandable structure.

How could one teach the child to multiply 3×32 ? If one were to buy three objects at 32 cents each, it would make the usual procedure more understandable if we thought $3 (30 + 2)$, then wrote $(3 \times 30) + (3 \times 2)$. In terms of our buying situation, we imagine that the 3 objects cost 30 cents each and then add on the extra two cents for each of 3 objects. Thus, $3 (30 + 2) = (3 \times 30) + (3 \times 2) = 90 + 6 = 96$. The fundamental principles employed in the above illustration are more clearly exhibited by writing $3 \times 32 = 3 (30 + 2) = (3 \times 30) + (3 \times 2) = (3 \times 3 \times 10) + (3 \times 2) = (9 \times 10) + 6 = 90 + 6 = 96$. Note how we employed the grouping and distributive principles for multiplication.

The distributive principle can be illustrated by writing $2 (5 + 6) = (2 \times 5) + (2 \times 6)$. Children use this principle without much thought when working such problems as: On Monday Tommy bought 12 apples at 5 cents each and on Tuesday he bought 8 more apples at 5 cents each. How much did Tommy spend for apples? The child may think: $\$.05 (12 + 8) = \$.05 \times 20 = \$1.00$ or he may think $\$.05 \times 12 + \$.05 \times 8 = \$.60 + \$.40 = \$1.00$. If he sees the problem both ways he recognizes the distributive principle. If he understands these two ways to work the problem, he is on his way to understanding this very important mathematical idea. He will come to understand it even

better when he studies algebra. In junior high school the distributive principle is employed when the perimeter (P) of a rectangle is written in two ways, namely, $P = 2a + 2b = 2(a + b)$.

The distributive principle is used three times when multiplying such numbers as 32×45 . Let us write them in the form $(3 \times 10 + 2)(4 \times 10 + 5)$ and see how this principle is used.

First distribute $3 \times 10 + 2$ over $4 \times 10 + 5$. There then results

$$(3 \times 10 + 2) 4 \times 10 + (3 \times 10 + 2) 5.$$

Now using the distributive principle twice more, we have

$$3 \times 10 \times 4 \times 10 + 2 \times 4 \times 10 + 3 \times 10 \times 5 + 2 \times 5.$$

Using the commutative principle several times, we can write

$$3 \times 4 \times 10 \times 10 + 2 \times 4 \times 10 + 3 \times 5 \times 10 + 2 \times 5.$$

Or by use of the associative principle

$$12 \times 100 + 8 \times 10 + 15 \times 10 + 10$$

$$\begin{array}{r} 1200 \\ 80 \\ 150 \\ 10 \\ \hline 1440 = 32 \times 45. \end{array}$$

Now this is a long process, but it is taking place when the child learns to multiply 32×45 . It is no wonder that teachers resort to saying, "This is the way you do it." But is this justifiable?

If we multiply 32×45 in the way you were taught, we obtain the separate products 1200, 80, 150 and 10 as in the above.

$$\begin{array}{r} 45 \\ 32 \\ \hline 10 - 2 \times 5 \\ 80 - 2 \times 40 \\ 150 - 30 \times 5 \\ 1200 - 30 \times 40 \\ \hline 1440 \end{array}$$

The result is the same as that obtained by our more familiar algorism.

We have now developed the five basic principles of our number system. New numbers (fractions), which will also be made to obey these principles, will be added later. Because we can make fractions obey these laws, we call them numbers even though they are made up of two of the numbers previously studied, that is, $\frac{2}{3}$ is a number pair made of a 2 and a 3.

We can now say that we have structured the number system for the elementary school. For the elementary school, numbers obey these five laws; anything that is constructed so as to obey these laws is called a number. This is a mathematical structure which children should encounter through well-structured physical experiences. It is now time to consider the perceptual experiences that lead the child to accept as numbers such other mathematical entities as common fractions.

PAIRS OF NUMBERS

There comes a time in the life of a child when some of the quantitative situations he faces cannot be adequately described in terms of one whole number. He is forced to learn how to work the pairs of numbers. Furthermore, he finds that there are many different physical situations which can be neatly described by the use of two natural numbers (whole numbers).

Rate Pairs

Suppose a child misses two words in the weekly assignment of ten spelling words. The teacher writes $\frac{2}{10}$ on the paper. This is not a fraction for the child. It merely says, "2 of the 10." Soon he learns that a record of $\frac{2}{10}$ is neither better nor worse than the record $\frac{1}{5}$.

Suppose John and his younger brother Jim do odd jobs on Saturdays for their neighbors. They agree that what they earn will be divided 3 for John and 2 for Jim ($\frac{2}{5}$). At the end of a day's work one can well imagine the boys sitting down and John drawing out 3 dimes from their total earnings while Jim draws out 2 dimes. They soon get tired of this and see that it is possible to

speed things up if John takes 6 dimes for Jim's 4 dimes. In essence, they have recognized that $\frac{3}{2} = \frac{6}{4}$.

Here we have the inception of the rate pair idea. It is only a short step for a teacher to structure a physical situation so that it is immediately apparent to the children that $\frac{3}{2} = \frac{6}{4} = \frac{9}{6} = \frac{12}{8} = \frac{21}{14} \dots$, and to show them that all of these are merely names for the same idea. (We will call 4 a name for a number and $\frac{3}{2}$ a rate pair.) From such experiences the child can derive an important mathematical principle. In any rate pair we may multiply both terms by the same number to obtain an equal rate pair, that is,

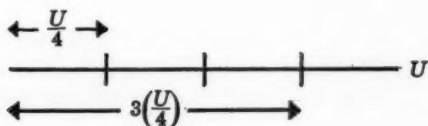
$$\frac{3}{2} = \frac{2 \times 3}{2 \times 2} = \frac{6}{4}.$$

Common Fractions

The above situation is structured in such a way that it fits the rate pair idea. It would be a mistake to use this situation to introduce the concept of fraction. It does not fit the fraction idea. In the above instance, we do not add the rate pairs. Because of this, it would be difficult to convince the child that we should call these pairs *numbers*. To introduce fractions, we must look for another situation, structured differently and more nearly like the *joining-groups* idea which the child has taken as his meaning of *add*. So another beginning is required. A measurement situation will serve our purposes admirably.

It frequently happens in life situations that we start with a unit and it is necessary to measure off a part of this unit. Call the unit U and let it be a line representing a piece of string or ribbon. To use U itself as a measure is impossible because the line to be measured is shorter than U . What now? Why, cut U into equal pieces (use 4 here) and use one of these pieces as a new unit.

Call this new unit $\frac{U}{4}$ for the present. (In the elementary school,



we could call it $\frac{1}{4}$, of course.) Now, it is possible to lay off $\frac{U}{4}$ three times along a line to be measured. We symbolize this by $3\frac{U}{4}$ (16:30-31). Normally, we call this $\frac{3}{4}U$ and say that the line is three-fourths of U long.

If $\frac{U}{4}$ were laid off on line L seven times we could write $7\frac{U}{4}$, but normally we write $\frac{7}{4}U$ and say that L is seven-fourths of U .



As long as we keep the same U it is obvious that $3\frac{U}{4} + 5\frac{U}{4} = 8\frac{U}{4}$. Written in a slightly more familiar way $\frac{3}{4}U + \frac{5}{4}U = \frac{8}{4}U$ and, still more familiar, $\frac{3}{4} + \frac{5}{4} = \frac{8}{4}$. Note, however, that we have omitted the U in the last equation.

To emphasize the role of U , consider this problem. There are two pies A and B as shown below.



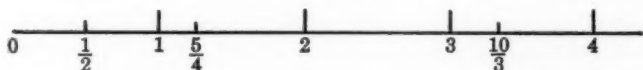
Does $\frac{1}{2}$ of a pie and $\frac{1}{4}$ of a pie equal $\frac{3}{4}$ of a pie? Only if the pies are the same size. Obviously, $\frac{1}{2}$ of Pie A + $\frac{1}{4}$ of Pie B is a meaningless statement if one wants to convey an idea of the quantity of pie. As a word pattern, it would be possible to say that $\frac{1}{2}$ of Pie A + $\frac{1}{4}$ of Pie B = $\frac{3}{4}$ of Pie X, but how big is Pie X? This is not at all satisfactory, so in arithmetic we adopt the convention that we are always considering pies of the same size. Then we don't even mention the pie and write $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$.

Now we have entities that can be combined just as the groups are. This gives the child a feeling that speaking of adding $\frac{1}{2}$ pie and $\frac{1}{4}$ pie to get $\frac{3}{4}$ pie is natural because we can represent it just

as we join groups. The physical actions are the same in both cases and this leads to easy acceptance. Of course, if the fractional parts of pies happen to have different denominators, it is necessary to teach the child to cut the pies into pieces of the same size and to symbolize this process with numerals.

Presenting a physical structure to fit a symbolism acceptable to mathematicians is decidedly unmathematical but the major problem at this stage is not so much a mathematical one as it is to show how we plan the instructional sequence to get the child to accept the fact that $\frac{1}{3} + \frac{1}{3}$ is the same as $\frac{2}{3} + \frac{2}{3}$. Experienced teachers have found that this is most easily done by setting up situations involving objects and manipulating them in such a way that the resulting combinations of objects and manipulations are structured to fit the mathematical symbolism.

To help the child place these number pairs into the system of numbers, he should be taught that they may be ordered. Here again it is necessary to look for a convenient picturing of a mathematical idea; the number line is this convenient device. In association with the principle that two fractions may be easily ordered



if they have the same denominator, the number line supplies *this worldly feeling* that it is all right to say $\frac{5}{4} > \frac{1}{2}$. The number line is a physically structured device which enables the child to accept readily the fact that $\frac{5}{4} > \frac{1}{2}$. Soon, however, he must cast aside the physical structure and deal entirely in symbols. This is the mark of maturity.

Space limitations prohibit our discussing in detail how to place decimal fractions on a number line. However, the generalization is easy and may be supplied by the reader.

We have now structured rate pair situations and fraction situations. The reader will note that the fraction number pairs have been placed in the realm of numbers by showing that they can be manipulated as numbers (added and multiplied) with the proper agreement on how to use numerators and denominators. Number pairs arising out of situations which divide a pile of apples in the

relation 2 to 3 are not so readily placed under the rules governing the fundamental operations of arithmetic. This thought leads us to investigate some of the common uses of number pairs as they occur in arithmetic. Percent comes to mind almost immediately.

Percents

Percents are really rate pairs. This is obvious if one recalls the thought pattern. What does 6% mean? Why 6 per 100, of course. The *per* may be interpreted as *for every* or *out of every*; for example, "you get \$6 *for every* \$100 the company grosses" or "you get 6 *out of every* 100 I earn." Both situations can be expressed by the term *6 per 100*.

Now the reader will have noticed that in rate pair situations (these include percents) there is no *U* as there was in the fraction situations. Here, then, is the difference between the physical structure of rate situations and the structure of fraction situations. Because of the absence of the *U*, it would be a mistake to begin instruction in percentage by defining 6% as .06 because .06 contains an implied *U* and 6% does not. But this is frequently the practice. As a number, .06 fits on a number line. It is a rational number (fraction). As normally used, 6% implies a base which is usually different from unity while .06 always has a *U* equal to one just as a fraction such as $\frac{3}{4}$ has a *U* equal to one. This lack of an understood *U* is a characteristic of the structure of a rate situation.

If children are taught decimal fractions in a number-line sense, or something equivalent to it (the number line has a *U*), it certainly must be confusing to be told early in the instructional sequence that 6% equals .06. From the physical structure point of view, 6% and .06 are entirely different. They should not be taught as one.

Now what has been the essential difficulty in the $6\% = .06$ situation? Basically, the machinery of arithmetic has overshadowed the learning sequence. Since, in the end, we want the child to write $6\% \text{ of } 16 = .06 \times 16$ we overlook the thought patterns behind these two groups of symbols. In other words, we use the wrong physical structure to illustrate the mathematical idea. The

fact that multiplying .06 by 400 yields the same answer as 6% of 400 is no argument for saying that these arise out of the same situation. For example, the fact that $3 \times 2 = 2 \times 3$ does not justify us in contending that the physical structure of 3 twos and 2 threes are the same. The thinking that fits percent situations is rate thinking. In initial learning stages the symbols $\frac{6}{100} = \frac{n}{400}$ fit the "6% of 400" situations better than $.06 \times 400$.

This same reasoning applies to rates; *2.54 cm. for every inch* should initially be thought of as $\frac{2.54}{1}$. Then changing 6 inches to

centimeters becomes $\frac{2.54}{1} = \frac{n}{6}$. In the same way, *6 apples for 40*

cents becomes $\frac{6}{40}$, and $\frac{6}{40} = \frac{n}{20}$ fits the structure if we ask, "How many for 20¢?"

SUMMARY

We have indicated that arranging situations so that the child can readily perceive essential structural features is a fundamental problem of instruction. These structural features must be appropriately symbolized and the results placed a more abstract structure—a mathematical structure. Good instruction produces good fits between the physically structured elements and the abstract principles which the child is to draw from the experience. An analysis of the situations commonly presented in arithmetic from this point of view is sorely needed. Basic structural inconsistencies can only make arithmetic hard to learn. This thought really leads us to conceive of learning as a level of the thinking process. The child is first introduced to an idea through well-constructed, physically-structured situations. Through recognizing and generalizing the elements of such situations, and after becoming thoroughly familiar with them, the child learns to deal with their symbols. (Maturity brings the ability to express ideas by means of and skill in manipulation of symbols without the need for thought.)

Years ago Judd (8:95-96) wrote:

Our analyses lead us to a recognition of several levels of thinking. The lowest level is the recognition of a concrete situation in which something is to be done with a group of things. The way in which the group is to be manipulated must be understood. Often the group is manipulated most readily by being translated into an equivalent group of tallies. There is a great variety of ways in which the rearrangements called for in arithmetic problems are to be made. Certain of these rearrangements are generically alike, and they may be classified together as additions. Others are generically alike in a totally different way and must be classified as subtractions. "Addition" is a word which can be used to describe a redistribution of concrete objects or a redistribution of tallies. The same can be said of "subtraction," "multiplication," and "division."

Early in their school lives pupils are called on to master as best they can this complex of more or less similar concrete situations and more or less systematic groupings of tallies. It is little wonder that many of them fail to understand the processes of arithmetical combination. One of the chief reasons for the failure of pupils in arithmetic is that general ideas are not explicitly taught. Teachers expect pupils to think abstractly and to generalize when they have had no adequate instruction in these difficult intellectual processes. To correct the present unsatisfactory condition, arithmetic must be understood with respect to its psychological complexities as well as with respect to its mathematical characteristics.

Judd speaks of situations being "generically alike." This chapter has used the term "structurally the same." Both terms refer to the same concept.

Mathematicians have structured the general ideas Judd was talking about. These ideas are very general. However, there is still need to organize the generically similar situations found in arithmetic into a consistent pattern. When this is done, arithmetic will make more sense.

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Guiding the Learner to Discover and Generalize

JOHN R. CLARK

ABOUT 40 YEARS AGO Dr. Charles Hubbard Judd, in an educational psychology course at the University of Chicago, reported on a significant learning experiment that involved throwing a javelin at an underwater target. Because of the refraction of light passing through media of different densities, the submerged target was not where it appeared to be. The participants in one group were familiar with the concept of refraction; members of the other group were not. The improvement in hitting the target made by the group unfamiliar with the concept was relatively slight, and was explained by trial and error. The improvement made by the group familiar with the concept was astounding. Judd's conclusion was that learning is strongly influenced by concepts or understandings.

To illustrate that discovery or generalization is dependent upon concepts, teachers often use the following sequence of integers,

asking, "Can you find the missing numbers?" Without the con-

2, 3, 5, 7, 11, 13, 17, ?, ?, ?

cept of prime number the responses are most often incorrect, even among persons of very high intelligence. Those persons who have the concept of prime numbers soon make a correct response, even though they try (and reject) concepts of constant difference, constant ratio, and repeated cyclical periods. We cannot generalize or see the pattern involved here without the strategic concepts of sequence and primes!

Another illustration is in order: What is the sum of any number of consecutive odd integers, beginning with 1? Readiness for the successful attack includes, as prerequisites, the concepts of sum, consecutive odd integers, and the square of a number, and the experimental attitude, "Let's take some instances and see what we can see."

With these prerequisites the learner may be expected to proceed somewhat as follows:

If the addends are:	The number of addends is:	Sum of addends is:
1 + 3	2	4
1 + 3 + 5	3	9
1 + 3 + 5 + 7	4	16
1 + 3 + 5 + 7 + 9	5	25

At this stage of the investigation, many learners at a maturity level of about 14 years will catch on, will see the relationship or sense the generalization, and will be able quickly to determine the sum for any number of such addends.

Pupils may verbalize the generalization in such ways as:

For 5 addends, a sum of 25; for 6 addends, a sum of 36; for 7 addends, a sum of 49; and so on.

The sum is the number of addends times that number.

The sum is the square of the number of addends.

I can't say it, but I can solve any of the problems.

The more symbolic statement of the generalization, $S = 1 +$

$3 + 5 + \cdots (2n - 1) = n^2$, requires a much higher maturity level.

In this chapter we assume agreement on the following aspects of arithmetic learning:

Pupils should discover and generalize. This is one of our basic objectives.

A generalization relates two or more concepts.

Pupils' generalizations vary in verbalization, symbolization, and generality.

The permanence and transferability of learning are increased by emphasis on generalizing.

A learner's success in problem-solving is dependent upon his possession (and recall) of generalizations which fit or are relevant to the "field" or situation under consideration.

The value to the learner is more in making a generalization than in memorizing it.

We now state in words and/or in mathematical symbols some of the great generalizations of arithmetic dealing with the science of operation, mensuration, and social (business) applications.

GENERALIZATIONS IN ARITHMETIC

1. The position of a digit in a numeral indicates its size or order (ones, tens, hundreds, etc.); the digit indicates the frequency of the order.

2. Changing the order of the addends does not affect the sum. Thus $A + B = B + A$. (The commutative law of addition.)

3. Addends may be grouped in any way without affecting the sum. Thus $A + B + C = (A + B) + C$, or $A + (B + C)$. (The associative law of addition.)

4. Addition and subtraction are opposite operations. If $A + B = C$, then $A = C - B$, and $B = C - A$. Also, if $A = C - B$, or $B = C - A$, then $A + B = C$.

5. The factors of a product are interchangeable. Thus $A \times B = B \times A$. (The commutative law of multiplication.)

6. The factors of a product may be taken (grouped) in any way. Thus $A \times B \times C = (A \times B) \times C = A \times (B \times C)$. (The associative law of multiplication.)

7. $N \times (A + B + C) = N \times A$, plus $N \times B$, plus $N \times C$. (The distributive law of multiplication.)

8. Multiplication and division are opposite operations. If $A \times B = C$, then $A = C \div B$, and $B = C \div A$. Also, if $A = C \div B$, or $B = C \div A$, then $A \times B = C$.

9. $(A + B + C) \div N = A \times \frac{1}{N} + B \times \frac{1}{N} + C \times \frac{1}{N}$, that is, each term of the dividend must be multiplied by the reciprocal of the divisor. (The distributive law of multiplication.)

10. A fraction times a number equals the product of the numerator of the fraction and the number, divided by the denominator, for example, $\frac{A}{B} \times C = \frac{A \times C}{B}$.

11. To multiply a product of several factors by a number, multiply only one of the factors by the number.

12. To divide a product of several factors by a divisor, divide only one of the factors by the divisor.

13. Dividend and divisor (or numerator and denominator) may be multiplied by or divided by any number other than zero without affecting the quotient.

SOME IMPORTANT GENERALIZATIONS RELATING TO GEOMETRY

1. The sum of the angles of a triangle is 180°
2. $A = b \times h$ (area of rectangle)
3. $A = s^2$ (area of square)
4. $A = \frac{b \times h}{2}$ (area of triangle)
5. $A = \pi \times r^2$ (area of circle)
6. $A = 4 \times \pi \times r^2$ (area of surface of sphere)
7. $V = l \times w \times h$ (volume of rectangular solid)
8. $V = \frac{4}{3} \times \pi \times r^3$ (volume of sphere)
9. If $AC = BC$ in $\triangle ABC$, then $\angle A = \angle B$ (Fig. 1)
10. If $\angle A = \angle B$ in $\triangle ABC$, then $AC = BC$ (Fig. 1)

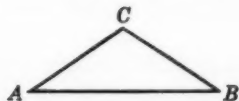


Figure 1

11. If $a^2 + b^2 = c^2$, then $\angle C = 90^\circ$ (Fig. 2)
 12. If $\angle C = 90^\circ$, then $a^2 + b^2 = c^2$ (Fig. 2)

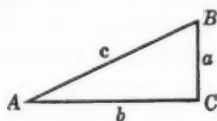


Figure 2

13. If $\angle A = \angle A'$ and $\angle B = \angle B'$, then $\angle C = \angle C'$ (Fig. 3)
 14. If $\angle A = \angle A'$ and $\angle B = \angle B'$, then (Fig. 3)

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$$

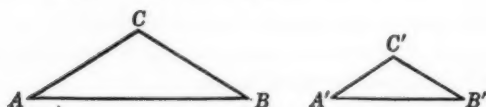


Figure 3

SOME IMPORTANT GENERALIZATIONS FROM BUSINESS PRACTICE

1. $c = n \times p$ (Cost of n articles at p cents each)
2. $C = r \times s$ (Commission C on sales s at rate r)
3. $d = r \times m$ (Discount on marked price m at rate r)
4. $i = r \times p$ (Interest on principal p for 1 year at rate r)
5. $p = s - c - o$ (Profit = selling price - cost price - overhead)
6. $p = r \times b$ (Percentage = rate \times base)
7. $C = g - p$ (Change = amount given - cost)

Admittedly, each of these lists of generalizations is incomplete. Some teachers may wish to expand the lists by adding new ones, or to make two or more generalizations from one or more of those in the lists.

Now we consider the problem of helping the learner arrive at some of the generalizations listed above. Ideally, we would like the learner to discover them, to create them. Through judicious

questioning and class discussion, with a minimum of telling, we should like them to be the learner's own conclusions or summary expressions of relationships being investigated. In the pages that follow, we shall employ a variety of procedures—questioning, inductive development, experiment, and the use of exposition—as we consider selections from these lists of generalizations.

Only teachers who understand the generalizations and who are familiar with the learner's arithmetic background and mental maturity can judge the learner's potential success in making a given generalization. Only such teachers can judge the degree of breadth and symbolism which should characterize his generalization.

ARITHMETIC

The Position of a Digit in a Numeral Indicates Its Size or Order

To develop this generalization, some teachers use empty cans, one for each order of digits, and sticks, as suggested in Figure 4.



Figure 4

The sticks in the hundreds' can (h) are interpreted as standing for 3 hundred; etc. The number represented in the drawing is 3 hundreds, 1 ten, 2 ones.

A somewhat more abstract scheme for studying the position of a digit in a numeral is the number frame, shown below.

h	t	o
3	2	1
2	0	3
4	5	0

We often hear teachers of second or third grade say, "Make your number frames. Mark the ones' column with an o, the tens column with a t, and the hundreds' column with an h." The teacher then dictates:

1. Write a digit 4 in hundreds' place, a digit 3 in tens' place, and a digit 6 in ones' place. Now read your number. What digit do you read first? (The digit in hundreds' place.)

2. Write on your number frame the number which means 5 hundreds, zero tens, 5 ones.

3. Could we write the number two hundred thirty-eight without using a number frame? How many digits are needed? What is its left hand digit? the middle digit? its right hand digit?

4. On this frame

	h	t	o
A			2
B		2	
C	2		

the 2 in Row A stands for ? ;

the 2 in Row B stands for ? ;

the 2 in Row C stands for ? .

After much practice, the learner will be able to make generalizations about the position of a digit in a numeral equivalent in meaning to:

A digit 2 in ones' place stands for 2 ones; in tens' place it stands for 2 tens, or 20; in hundreds' place it stands for 2 hundreds, or 200.

If you imagine any digit, say 5, being moved from ones' place to tens' place, its worth is then increased to 5 tens, or 50; if it is then moved to hundreds' place, its worth is then increased to 5 hundreds, or 500.

In a numeral such as 777, each 7 is worth 10 times the 7 to its right, and $\frac{1}{10}$ the 7 to its left.

The digits of a numeral have place value. The places are called ones' place, tens' place, hundreds' place, and so on.

The place or position of a digit shows its value.

A similar development of the positional value of digits to the right of ones' place (on a higher grade level) will lead to generalizations equivalent to:

The decimal point identifies the ones' place, and it separates the integral and fractional parts.

The first place to the right of ones' place is tenths' place, etc.

In a numeral such as 11.1, each 1 is worth 10 times the 1 to its right, and one-tenth of the 1 to its left.

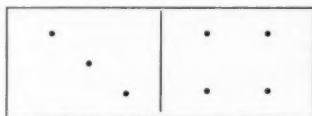
The Commutative Law of Addition: $A + B = B + A$

The use of exercises such as the following should enable the learner to arrive at this basic generalization.

1. In Picture A you see that $4 + 3 = \underline{\quad? \quad}$. In Picture B you see that $3 + 4 = \underline{\quad? \quad}$.



A



B

2. Do you think that $8 + 5$ are as many as $5 + 8$?

3. In Box C you see that the sum of 37 and 14 is $\underline{\quad? \quad}$. Box D asks "What is the sum of 14 and 37?" How does Box C help you tell the sum of 14 and 37, without actually doing the addition?

37
+14
—
51

C

14
+37
—
?

D

Which of the statements in exercises 4 through 7 do you think are true?

4. The sum of 27 and 18 is equal to the sum of 18 and 27.

5. The direction (up or down with respect to vertical addition, to the right or to the left with respect to horizontal addition) of the adding doesn't change the sum.

6. If one addend is A and the other addend is B , then $A + B = B + A$.

7. If A is larger than B , then $A + B$ is larger than $B + A$.

8. Does $A + B = B + A$

when A is a fraction and B is a whole number?

when A is a three-digit number and B is a two-digit number?

when A is a whole number and B is a decimal?

9. Which of these statements do you prefer? Why?

The addends may change places without changing the sum.

The order of the addends makes no difference in the sum.

If A and B are numbers, any kind of number that we have studied, then $A + B = B + A$.

The technical name for the generalization $A + B = B + A$ is the *commutative law of addition*. This and other such technical names of laws of operation should seldom, if at all, be used with elementary school pupils.

The Commutative Law of Multiplication

For the development of this generalization, exercises such as the following may be used.

1. This drawing shows that 2 fours are as many as ____?____ twos.

: : : :
: : : :

2. In the statement $2 \times 4 = 8$, the 2 and the 4 are the *factors* and the 8 is the *product*. What are the factors and the product in the statement $4 \times 2 = 8$?

3. If you know that $7 \times 12 = 84$, would you then know that $12 \times 7 = 84$? Examine this check.

12	7
$\times 7$	$\times 12$
84	14
	$\underline{7}$
	84

4. If you know that $12 \times 15 = 180$, would you need to do a multiplication to find 15×12 ? Check your answer.

Which of the statements in exercises 5 through 10 do you think are true?

5. Number line (a) in Figure 5 shows that $4 \times 6 = 24$, and number line (b) shows that $6 \times 4 = 24$.

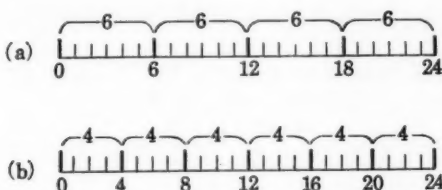


Figure 5

6. The factors may change places without changing the product.

7. If the factors are A and B , and the product is C , then $A \times B = C$, and $B \times A = C$.

8. If A is larger than B , then $A \times B$ is larger than $B \times A$.

9. If A is a mixed number, and B is a whole number, then $A \times B = B \times A$.

10. If A is zero and B is not zero, then $A \times B = B \times A$.

11. When would it be helpful to make use of $A \times B = B \times A$? The technical name for $A \times B = B \times A$ is the *commutative law of multiplication*.

12. Write a sentence using the words: *product*, *factors*, *order of*.

The Associative Law of Addition

The guidance provided by exercises such as the following should enable the learner to make the generalization and recognize its usefulness.

1. Which of these ways of thinking, (a) or (b) below, would you use to find the sum of these addends:

13

14

25

(a) $27 + 25$ (the 27 is the sum of 13 and 14)?

(b) $13 + 39$ (the 39 is the sum of 14 and 25)?

2. In exercise 1 the addends were regrouped in these ways:

$$(a) (13 + 14) + 25$$

$$(b) 13 + (14 + 25)$$

Does the way in which the addends were regrouped affect their sum?

3. Which of these regroupings of the addends in $6 + 2 + 4 + 18 + 9$ makes the addition easier for you?

$$(a) 6 + (2 + 4) + (18 + 9)$$

$$(b) (6 + 4) + (18 + 2) + 9$$

4. In what ways are these methods of finding the sum of 27 and 18 alike? different?

$$(a) 27$$

$$\underline{18}$$

$$30 (20 + 10)$$

$$\underline{15} (7 + 8)$$

$$45$$

$$(b) 27$$

$$\underline{18}$$

$$15 (7 + 8)$$

$$\underline{30} (20 + 10)$$

$$45$$

$$(c) \overset{1}{27}$$

$$\underline{18}$$

$$45$$

5. What conclusion about addition can you make from your study of exercise 4?

6. Complete: You can find the sum of some of the addends, then find the sum of the others, and then . . .

7. If a is 27, b is 6, and c is 3, which of these ways of regrouping the addends would you use?

$$(a + c) + b, \quad \text{or} \quad (b + c) + a, \quad \text{or} \quad (a + b) + c$$

8. What conclusion can you draw from thinking about the regrouping of addends?

9. Write a sentence about regrouping of addends, using the words: *addends*, *sum*, *regrouped*.

The Relationship Between Addition and Subtraction

The elementary school learner is too immature to appreciate the mathematician's statement, "subtraction is *defined* as the opposite of addition," or " $c - a = b$ if the sum of a and b is c ," or "the difference between a minuend (c) and a subtrahend (a) is the number which added to a gives a sum of c ." These statements are too abstract, too remote from the learner's experience. Hence a slow, planned, sequential development of these concepts and relationships is essential.

To guide the learner to discover or rediscover the relationship between addition and subtraction, teachers employ learning situations such as:

1. Tom saved 17¢ one week and 28¢ the next week. During the two weeks he saved the sum of 17¢ and 28¢, or ____ cents.

$$\begin{array}{r} 17¢ \text{ addend} \\ + 28¢ \text{ addend} \\ \hline 45¢ \text{ sum} \end{array}$$

2. Identify the addends and the sum in exercise 1.

3. After Tom had saved 45¢, he bought a ball point pen for 28¢. Then he had 45¢ - 28¢, or ____ cents.

$$\begin{array}{r} 45¢ \text{ minuend} \\ - 28¢ \text{ subtrahend} \\ \hline 17¢ \text{ difference or remainder} \end{array}$$

4. Identify the minuend, subtrahend and remainder in Tom's problem (exercise 3).

5. If $17 + 28 = 45$, then $45 - 28 = \underline{\quad? \quad}$, and $45 - 17 = \underline{\quad? \quad}$.

6. In $a + b = c$, the addends are ____ and ____; the sum is ____.

7. In $a = c - b$, the minuend is ____; the subtrahend is ____; and the remainder or difference is ____.

Use exercise 5 to show that:

The sum (in addition) corresponds to the minuend (in subtraction).

One of the addends (in addition) becomes a subtrahend (in subtraction).

The other addend becomes a remainder.

8. If you know that $37 + 28 = 65$, which of these facts do you also know, without doing any computing?

$$28 + 37 = 65 \quad 65 - 28 = 37 \quad 65 - 37 = 28$$

9. The four facts in exercise 8 are often called the "28, 37, 65" family of number facts. Why is that a good name for them?

10. What are the four facts in each of these families?

- (a) the "64, 36, 100" family (b) the "87½, 12½, 100" family

11. If $50 - 18 = 32$, which of these statements do you believe to be true?

- (a) $50 - 32 = 18$ (b) $50 = 18 + 32$

12. Given that $a + b = c$, does it then follow that

$$a = c - b? \quad b = c - a? \quad a \text{ is larger than } b?$$

The Associative Law of Multiplication

Exercises such as the following should enable the learner to make this generalization, and appreciate its usefulness.

1. Which of these ways of finding $3 \times 10 \times 27$ do you prefer? Why?

- (a) $10 \times 27 = 270$ (b) $3 \times 10 = 30$
 $3 \times 270 = 810$ $30 \times 27 = 810$

2. In $3 \times 10 \times 27$, the factors are 3, 10, and ____?

3. If a is 3, b is 10, and c is 27, then $a \times b \times c =$ ____?

4. Which way of doing the multiplication $4 \times 2\frac{1}{2} \times 6\frac{1}{2}$ do you prefer? Why?

- (a) $4 \times 2\frac{1}{2} = 10$ (b) $2\frac{1}{2} \times 6\frac{1}{2} = 15\frac{1}{2}$
 $10 \times 6\frac{1}{2} = 62\frac{1}{2}$ $4 \times 15\frac{1}{2} = 62\frac{1}{2}$

5. In $4 \times 2\frac{1}{2} \times 6\frac{1}{2}$, the factors are ____, ____, and ____?

6. If a is 4, b is $2\frac{1}{2}$ and c is $6\frac{1}{2}$, then the product of $a \times b \times c$ is ____?

7. If there are three or more factors in a product, which two of them would you use first?

8. If the factors are represented by a , b , and c , and the product is represented by B , do you think that the factors could be taken in *any order*, without affecting the product? Use exercise 4 to illustrate your answer.

9. Write a statement using the words *product*, *factors*, *grouping*.

10. What do you think this statement means?

$$a \times b \times c = (a \times b) \times c = a \times (b \times c)$$

11. Jane could multiply by one-digit multipliers, but not by two-digit multipliers. She needed to find 12×128 . She thought "I'll use 4 and 3, factors of 12. Then I'll have $(3 \times 128) \times 4$, or ____."

12. Find 12×128 by using 6 and 2 as factors of 12.

13. Try Jane's idea for finding 15×107 ; 36×19 ; 42×13 .

The Distributive Law of Multiplication

From a consideration of exercises such as the following, the learner should be able to make the generalization: $a(b + c) = ab + ac$.

1. How are these methods of finding 8×45 alike? different?

(a) 45

$\times 8$

320 (8×40)

40 (8×5)

360

(b) 45

$\times 8$

40 (8×5)

320 (8×40)

360

(c) 45

$\times 8$

360

2. In exercise 1 the multiplicand 45 was separated into two terms or parts, 40 and 5 ($40 + 5 = \underline{\quad? \quad}$). Each term was multiplied by ____?

3. Do you think that the multiplicand in exercise 1 could be separated into the terms 20, 20, and 5? Does $20 + 20 + 5 = 45$? Does 8×20 , plus 8×20 , plus 8×5 , equal 360?

4. Do you think that the multiplicand in exercise 1 could be separated into the terms 30, 10, and 5? Does $30 + 10 + 5 = 45$? Does 8×30 , plus 8×10 , plus 8×5 , equal 360?

5. The statement $6 \times (30 + 5)$ means that each of the terms within the parentheses is to be multiplied by 6; $6 \times 30 = 180$, and $6 \times 5 = 30$, so $6 \times (30 + 5) = 180 + 30$, or ____?

6. What does the statement $8 \times (100 + 20 + 7)$ mean? Does 8×127 equal $8 \times (100 + 20 + 7)$?

7. Do you think that $a \times (b + c + d)$ equals $a \times b + c + d$, or does it equal $a \times b + a \times c + a \times d$? (Each number within the parenthesis is to be multiplied by ____?)

8. Which of these statements do you believe to be true?

(a) $12 \times 15 = 12 \times (10 + 5)$

(b) $12 \times 15 = 12 \times (5 + 10)$

$$(c) 12 \times 15 = 10 \times (15) + 2 \times (15)$$

$$(d) 12 \times 15 = 12 \times (5) + 12 \times (10)$$

9. The technical name for the principle of multiplication illustrated above is the *distributive law of multiplication*.

10. The distributive law of multiplication, when applied to finding 6×254 gives

$$(a) 6 \times (200) + 6 \times (50) + 6 \times (4), \text{ which equals } 1524, \text{ or}$$

(b) the familiar written form:

$$\begin{array}{r} 254 \\ \times 6 \\ \hline 1524 \end{array}$$

The Relationship Between Multiplication and Division

Division is the Opposite of Multiplication. The perfectly valid statement that division is the opposite of multiplication would be meaningless to the pupil in his early stages of the study of division. Prerequisites for comprehension of the opposite operation are the concepts of multiplier, multiplicand, product, dividend, divisor, and quotient.

The thinking illustrated in a development such as the following leads to the generalization that *division is the opposite of multiplication*.

16	multiplicand
$\times 12$	multiplier
32	
160	
192	product

A

	divisor
16	← quotient
12)192	← dividend
12	
72	
72	

B

1. In box A the product is ____?____; in box B the dividend is ____?____.
2. In box A the multiplier is ____?____; in box B the divisor is ____?____.
3. In box A the multiplicand is ____?____; in box B the quotient is ____?____.

multiplication, dividend, divisor, reciprocal, and quotient. Much experience in finding quotients by using number scales, counting the number of times the measuring unit (the divisor) is contained in the magnitude being measured (the dividend), and in using division algorithms is essential preparation for discovery and appreciation of the generalization.

Exercises such as the following are helpful in building the concept of reciprocal numbers.

1. What is the product of $\frac{2}{3}$ and $\frac{3}{2}$? of $\frac{3}{4}$ and $\frac{4}{3}$? of $\frac{5}{6}$ and $\frac{6}{5}$? of $\frac{7}{8}$ and $\frac{8}{7}$?

Two numbers whose product is 1 are called reciprocals, or reciprocal numbers; each number is the reciprocal of the other.

2. Are $\frac{2}{3}$ and $\frac{3}{2}$ reciprocals? Why?
3. Is $\frac{3}{4}$ the reciprocal of $\frac{4}{3}$? Why?
4. Is 3 the reciprocal of $\frac{1}{3}$? Is $\frac{1}{3}$ the reciprocal of 3?
5. Write the reciprocal of 4, $\frac{1}{2}$, $\frac{5}{8}$, $\frac{7}{9}$, 10, $\frac{17}{10}$, $\frac{3}{8}$, $\frac{8}{3}$.

At this point the learner should recall (or relearn) a previously developed generalization: *Either of two factors of a product equals the product divided by the other factor.*

Exercises such as the following may be used for this purpose:

1. The product of two factors is 81; one of the factors is 27; so the other factor is ____?
2. The quotient and the divisor are factors of the ____?

Which of the statements in exercises 3 through 7 do you believe to be true?

3. A dividend is the product of its two factors.
4. Any number (other than zero) times its reciprocal equals 1.
5. If you know that $\frac{2}{3} \times \frac{3}{2} = 1$, then you know that $1 \div \frac{2}{3} = \frac{3}{2}$, and that $1 \div \frac{3}{2} = \frac{2}{3}$.
6. The quotient of $18 \div 2$ equals the dividend (18) times the reciprocal of the divisor ($\frac{1}{2}$).
7. The quotient of $6 \div \frac{2}{3}$ equals the dividend (6) times the reciprocal of the divisor ($\frac{3}{2}$).

It is unnecessary here to point out that the dividend may be an integer, a fraction, a mixed number, or a decimal, and that the only restriction upon the divisor is that it may not be zero.

Hence the wide application of the generalization: *The quotient equals the dividend times the reciprocal of the divisor.*

It now becomes obvious that division is based upon the distributive law of multiplication. Thus

$$\begin{aligned}\frac{843}{4} &= (800 \times \frac{1}{4}) + (40 \times \frac{1}{4}) + (3 \times \frac{1}{4}) \\ &= 200 + 10 + \frac{3}{4} \\ &= 210\frac{3}{4}.\end{aligned}$$

Division: Separating the Dividend into Convenient Parts, and Multiplying Each Part by the Reciprocal of the Divisor

This basic generalization is promoted by thinking such as the following:

1. Here are three methods of dividing 175 by 4:

$$\begin{aligned}\text{(a)} \quad \frac{175}{4} &= (100 \times \frac{1}{4}) + (40 \times \frac{1}{4}) + (35 \times \frac{1}{4}) \\ &= \quad 25 \quad + \quad 10 \quad + \quad 8\frac{3}{4}\end{aligned}$$

$$\begin{aligned}\text{(b)} \quad \frac{175}{4} &= (160 \times \frac{1}{4}) + (15 \times \frac{1}{4}) \\ &= \quad 40 \quad + \quad 3\frac{3}{4}\end{aligned}$$

$$\begin{aligned}\text{(c)} \quad \frac{175}{4} &= (40 \times \frac{1}{4}) + (40 \times \frac{1}{4}) + (40 \times \frac{1}{4}) + (40 \times \frac{1}{4}) + (15 \times \frac{1}{4}) \\ &= \quad 10 \quad + \quad 10 \quad + \quad 10 \quad + \quad 10 \quad + \quad 3\frac{3}{4}\end{aligned}$$

2. See if you can think of another method for dividing 175 by 4.
3. Which of the three methods in exercise 1 requires the least work?
4. In (a) the dividend was separated into ? terms or parts.
5. In (b) the dividend was separated into ? terms or parts.
6. In which method was the dividend separated into the smallest number of parts? the largest number?
7. Which of the parts in (a) are multiples of 4? in (b)? in (c)?
8. Is it more convenient to separate the dividend, 175, into smaller multiples or into larger multiples of 4? Why?
9. To divide 175 by 5, which of these ways of separating the dividend into parts would you choose? Why?

$$(a) 5\overline{)175} = 5\overline{)150} + 5\overline{)25}$$

$$(b) 5\overline{)175} = 5\overline{)100} + 5\overline{)50} + 5\overline{)25}$$

$$(c) 5\overline{)175} = 5\overline{)50} + 5\overline{)50} + 5\overline{)50} + 5\overline{)25}$$

10. To divide 175 by 6, which of these ways of separating the dividend into parts would you choose? Why?

$$(a) 6\overline{)175} = 6\overline{)60} + 6\overline{)60} + 6\overline{)55}$$

$$(b) 6\overline{)175} = 6\overline{)120} + 6\overline{)55}$$

$$(c) 6\overline{)175} = 6\overline{)100} + 6\overline{)75}$$

11. Which of the ways in exercise 10 is least helpful? Why?

12. To divide $x + y + z$ by d , would you

(a) divide only parts of the dividend that are multiples of d by d ? or

(b) divide each part of the dividend by d ?

13. Which of these statements of the preceding principle do you prefer? Why?

$$(a) 2\overline{)156} = 2\overline{)100} + 2\overline{)50} + 2\overline{)6}$$

(b) Separate the dividend into any convenient parts; then divide each part by the divisor, and find the sum of the partial quotients.

The above generalization is obviously an implication of the distributive law of multiplication.

Fraction Multipliers

TEACHER: I want to see whether you pupils can discover a way to find $\frac{2}{3} \times 20$. You see that the multiplier is a fraction. What do you think?

JANE: I would change the example to $20 \times \frac{2}{3}$. We know how to repeat the $\frac{2}{3}$ twenty times. The product is $\frac{20 \times 2}{3}$ or $\frac{40}{3}$ or $13\frac{1}{3}$.

BILL: I think Jane is right. We know that multipliers and multiplicands may change places.

TOM: Isn't there some other good way to think about it?

TEACHER: Could we fit it into a multiplication table, like this:

$$54 \times 20 = 1080$$

$$18 \times 20 = 360$$

$$6 \times 20 = 120$$

$$2 \times 20 = 40$$

What is happening to the multipliers? Each one is what part of the previous one?

CLASS: Each is one-third of the previous one.

TEACHER: What is happening to the products? Each one is what part of the previous one?

CLASS: Each is one-third of the previous one.

TEACHER: Let's extend the table. What would be the next multiplier?

TOM: Oh yes. It would be $\frac{1}{3}$ of 2, or $\frac{2}{3}$. The next product would be $\frac{1}{3}$ of 40, or $13\frac{1}{3}$. That's just what we got before.

TEACHER: I like both ways. But isn't Jane's way much easier? She didn't have to make a table.

Teacher then gives the class practice, such as the following

$$\frac{3}{4} \times 20 \quad \frac{3}{4} \times 15 \quad \frac{4}{5} \times 30 \quad \frac{8}{9} \times 24$$

$$\frac{3}{4} \times 29 \quad \frac{3}{4} \times 21 \quad \frac{3}{4} \times 22 \quad \frac{3}{4} \times 23$$

Now we would expect further generalization, to include:

$$\frac{a}{b} \times c = c \times \frac{a}{b}, \quad \frac{a}{b} \times c = \frac{a \times c}{b}, \quad \text{and} \quad \frac{a}{b} \times c = a \times \frac{c}{b}$$

This thinking makes it meaningful to say $\frac{3}{4} \times 20$ equals $\frac{3}{4}$ of 20.

Multiplying $f_1 \times f_2 \times f_3 \times \dots$ by N

It is often convenient to employ the generalization: *To multiply the product of several factors by a number, multiply any one of the factors by the number.* This generalization may be developed using the following procedures.

PROBLEM: To find the product of $3 \times 8 \times 12\frac{1}{2}$ multiplied by 4.

Class discussion of the problem may be expected to bring

forth a number of suggestions, including:

1. Multiply each factor by 4, giving a product of $12 \times 32 \times 50$, or 19,200.
2. Multiply the factors, giving $3 \times 8 \times 12\frac{1}{2}$, or 300; then multiply 300 by 4, giving 1200.
3. Multiply only the first factor by 4, giving $12 \times 8 \times 12\frac{1}{2}$, or 1200.
4. Multiply only the second factor by 4, giving $3 \times 32 \times 12\frac{1}{2}$, or 1200.
5. Multiply only the third factor by 4, giving $3 \times 8 \times 50$, or 1200.

Evaluation of the procedures (by the learners) should result in the rejection of 1, acceptance of the others, with a preference for 3, 4, and 5.

Further investigation of the problem, employing such examples as the following

$$(a) 4(3 \times 5 \times 25)$$

$$(b) 6(7 \times 4 \times 8)$$

$$(c) 8(5 \times 6 \times 7)$$

should deepen the learner's insight, and enable him to formulate the generalization. Expressed as a number sentence, the generalization may be

$$N \times (a \times b \times c) = (N \times a) \times b \times c, \quad \text{or} \quad (N \times b) \times a \times c, \\ \text{or} \quad (N \times c) \times a \times b.$$

Obviously this generalization employs the commutative and the associative laws of multiplication.

Dividing $f_1 \times f_2 \times f_3 \times \dots$ by N

It is often convenient to employ the generalization: *To divide the product of several factors by a divisor, divide any one of the factors by the divisor.* The development of this generalization often proceeds as follows:

PROBLEM: Find the quotient of $8 \times 12 \times 20$ divided by 4.

Class discussion of the problem may be expected to bring forth a number of suggestions, including:

1. Divide each factor by 4, giving a quotient of $2 \times 3 \times 5$, or 30.

2. Multiply the factors, giving 1920; divide the 1920 by 4, giving 480.

3. Divide only the first factor by 4, giving $2 \times 12 \times 20$, or 480.

4. Divide only the second factor by 4, giving $8 \times 3 \times 20$, or 480.

5. Divide only the third factor by 4, giving $8 \times 12 \times 5$, or 480.

Evaluation of the procedures by the learners should result in the rejection of 1, and in the acceptance of 2, 3, 4, and 5, with a preference for 3, 4, and 5.

Further consideration of the problem, employing such examples as the following

$$(a) \overline{4)8 \times 15 \times 21}$$

$$(b) \overline{4)7 \times 12 \times 19}$$

$$(c) \overline{4)7 \times 11 \times 20}$$

should bring forth a verbalization equivalent to: *To divide a product of several factors by a divisor, divide only one factor, preferably one that is a multiple of the divisor.*

This generalization obviously follows from the associative law of multiplication, as illustrated below.

$$\begin{aligned} \frac{8 \times 15 \times 21}{4} &= (8 \times 15 \times 21) \times \frac{1}{4} \\ &= \left(8 \times \frac{1}{4}\right) \times (15 \times 21), \text{ etc.} \end{aligned}$$

Expressed in number sentences, the generalization may be:

$$\frac{a \times b}{c} = \frac{a}{c} \times b, \text{ or } a \times \frac{b}{c}.$$

Equivalent Quotients: $\frac{a}{b} = \frac{a \times n}{b \times n} = \frac{a \div n}{b \div n}$

This generalization has often been called the *fundamental principle of division*. By this principle we may change the form or the name of a fraction; thus these symbols all stand for the same number: $\frac{2}{3}, \frac{4}{6}, \frac{12}{18}, \frac{16}{24}$. Frequently we use the principle to

replace non-integral by integral divisors, as in

$$(a) \frac{10}{2.5} = \frac{10 \times 10}{10 \times 2.5} = \frac{100}{25}$$

$$(b) \frac{3\frac{1}{2}}{2\frac{1}{4}} = \frac{4 \times 3\frac{1}{2}}{4 \times 2\frac{1}{4}} = \frac{14}{9}$$

$$(c) \frac{\frac{1}{2}}{\frac{1}{3}} = \frac{6 \times \frac{1}{2}}{6 \times \frac{1}{3}} = \frac{3}{2}$$

Guidance in discovering the generalization is facilitated by procedures such as:

1. Do you think the quotient of 60 divided by 20 is equal to the quotient of $(60 + 1)$ divided by $(20 + 1)$? of $(60 + 2)$ divided by $(20 + 2)$? of $(60 + 10)$ divided by $(20 + 10)$?

In general, would you say that quotients are changed, or unchanged, when the same number is added to both the dividend and the divisor?

2. Do you think that the quotient of 60 divided by 20 is equal to the quotient of $(60 - 10)$ divided by $(20 - 10)$? of $(60 - 15)$ divided by $(20 - 15)$?

In general, are quotients changed, or unchanged, when the same number is subtracted from the dividend and the divisor?

3. Do you think that the quotient of 60 divided by 20 is equal to the quotient of (2×60) divided by (2×20) ? of (3×60) divided by (3×20) ? of (10×60) divided by (10×20) ? What general principle is illustrated here?

4. Do you think that the quotient of 60 divided by 20 is equal to the quotient of $(60 \div 2)$ divided by $(20 \div 2)$? of $(60 \div 5)$ divided by $(20 \div 5)$? of $(60 \div 20)$ divided by $(20 \div 20)$?

What general principle do these examples illustrate?

5. Which of these statements about equal quotients do you prefer?

(a) The quotient is unchanged when the dividend and the divisor are multiplied or divided by the same number.

(b) Using a and b for the numbers in the dividend and the divisor, and n for any number (other than zero).

$$\frac{a}{b} = \frac{n \times a}{n \times b} = \frac{a \div n}{b \div n}$$

- (c) It is permissible to multiply both dividend and divisor by the same number.
- (d) The numerator (dividend) and the denominator (divisor) may be multiplied or divided by the same number (other than zero) without changing the fraction.

GEOMETRY

Discovering the Sum of the Angles of a Triangle

PROBLEM: How would you find the sum of the angles of a triangle?

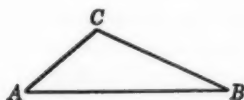


Figure 6

We may assume that an investigation would be launched, in which each member of the class draws any triangle he chooses, measures its angles, and finds their sum.

The recording of the data is made in the manner shown below.

	$\angle A$	$\angle B$	$\angle C$	Sum
Bill	55°	60°	64°	179°
Joe	49°	71°	61°	181°
Ann	71°	55°	54°	180°

Study of the data should lead to:

The triangles must not have been alike, because the sizes of the angles were so different.

The sums in all of the triangles seem to be close to 180° .

The average of all of our sums is 179.4° .

TEACHER: If we had been more careful in our measurements, do you think that the average sum would have been even closer to 180° ?

Doubtless the class would repeat the investigation, each pupil making a different, but larger, triangle. Should the second average

be ever so slightly closer to 180° , the conviction of the pupils would be deepened.

At this point the teacher would be wise to say: "Your results are very good. Mathematicians know that the sum of the angles of a triangle is 180° . They don't even have to do any measuring. But shall we now agree that in any triangle ABC, $\angle A + \angle B + \angle C = 180^\circ$?"

Discovering that a Triangle whose sides have the ratio of 3 to 4 to 5 is a right triangle

After having learned how to construct a triangle whose sides are equal to three given segments, a pupil in the class remarked, "I read that the Egyptians made right angles by using triangles the length of whose sides have the ratio of 3 to 4 to 5. They believed that the angle opposite the largest side is always a right angle. Could we test their idea?"

Under the teacher's guidance an experiment was planned to test the hypothesis. Each pupil chose a unit segment, constructed a triangle whose sides were 3, 4, and 5 units in length, and measured the angle opposite the 5-unit side. When this was done, the average (mean) result for the size of the angle was very close to 90° , and so the class decided that the angle was very close to, if not exactly, a right angle.



Figure 7

The teacher, knowing that triangles whose sides have certain other ratios, (for example, 5 to 12 to 13) are right triangles, asked the question, "Do you think that there are triangles other than those whose sides have the ratio of 3 to 4 to 5 that are also right triangles? For example, what about triangles whose sides have the ratio of 5 to 12 to 13?"

A similar investigation convinced the class that such triangles also are right triangles.

We may speculate that the teacher extended the investigation to triangles having sides, a , b , and c , such that $a^2 + b^2 = c^2$, thus arriving at the generalization: *A triangle whose sides a , b , and c are such that $a^2 + b^2 = c^2$ is a right triangle.*

At this point we should consider the difference between *demonstration* (showing and telling) and *discovery*. The writer is currently observing a television course in college physics in which the instructor employs the demonstration technique of teaching generalizations. In spite of the excellence of the demonstration (materials and exposition), the absence of pupil experimentation, class discussion, and questioning seem to make it difficult to remember, to speculate, to organize, and to generalize. It appears that there is time only for listening and looking.

APPLICATIONS IN DAILY LIFE

Thus far we have considered purely mathematical generalizations, those relating directly to the fundamental operations and to geometry. Now we come to the applications of arithmetic in daily life. Illustrative of these are the relationships among:

1. Profit, p (in dollars); selling price, s (in dollars); and total cost, c (in dollars). In number sentences the relationship may be expressed in these generalizations:

$$p = s - c, \text{ or } s = p + c, \text{ or } c = s - p.$$

2. Commission, c ; rate of commission, r ; and sales, s . The relationship among these concepts may be expressed in these generalizations:

$$c = r \times s, \text{ or } s = \frac{c}{r}, \text{ or } r = \frac{c}{s}.$$

3. Discount, d ; rate of discount, r ; and marked price, m . The relationship among these concepts leads to:

$$d = r \times m, \text{ or } r = \frac{d}{m}, \text{ or } m = \frac{d}{r}.$$

4. Rate of interest, r ; principle, p ; time in years, t ; interest, i .

$$i = r \times p \times t \quad p = \frac{i}{r \times t}$$

$$r = \frac{i}{p \times t}$$

$$t = \frac{i}{r \times p}$$

5. Number of square feet in area of rectangle, A ; number of feet in length, l ; number of feet in width, w .

$$A = l \times w \quad l = \frac{A}{w} \quad w = \frac{A}{l}$$

6. Number of units in perimeter of rectangle, p ; number of units in length of rectangle, l ; and number of units in width of rectangle, w .

$$p = 2l + 2w, \text{ or } l = \frac{p - 2w}{2}, \text{ or } w = \frac{p - 2l}{2}$$

7. If m represents the number of dollars you give the merchant, c represents the cost in dollars of what you buy, and r represents the amount of change you get, we then have the following generalizations:

$$m - c = r, \text{ or } m = c + r, \text{ or } c = m - r.$$

There are many such generalizations (see page 66). Pupils who have thought out many of these generalizations are more resourceful in problem solving.

Extending or Broadening Generalizations

The usefulness of a generalization is determined by the range of its applications. To illustrate:

1. In Box A we are finding the commission on sales of \$200, with a rate of commission of 5%. The generalization employed is *commission equals rate times sales*. The factors are the number of dollars of sale, and the rate (in hundredths); the product is \$10.00, the commission.

$$\begin{array}{r} \$200 \text{ (sales)} \\ \times .05 \text{ (rate of commission)} \\ \hline \$10.00 \text{ (commission)} \end{array}$$

A

2. In Box B we are finding the discount on an article marked to sell for \$200, which sold at a discount of 5%. What are the factors? What is the product?

$$\begin{array}{r} \$200 \text{ (marked price)} \\ \times .05 \text{ (rate of discount)} \\ \hline \$10.00 \text{ (discount)} \end{array}$$

B

3. In Box C, what problem is being solved? What are the factors? What is the product?

$$\begin{array}{r} \$200 \text{ (principal)} \\ \times .05 \text{ (rate of interest for 1 year)} \\ \hline \$10.00 \text{ (interest for 1 year)} \end{array}$$

C

4. How (in what respect) are the three solutions of the above problems alike and how are they different?

5. Look at the work in Box D. What are the factors? What is the product?

$$\begin{array}{r} 200 \text{ (base)} \\ \times .05 \text{ (rate)} \\ \hline 10.00 \text{ (percentage)} \end{array}$$

D

The *base* in Box D corresponds to what ideas in the other boxes? The *percentage* corresponds to what ideas in the other boxes?

6. Is the thinking in Box D more general, or less general, than the thinking in Box A? in Box B? in Box C? Why?

There is a continuing controversy concerning the use of the $p = r \times b$ generalization in the solution of problems illustrated in Boxes A, B, and C. Our *a priori* judgment is that the $p = br$ generalization should be developed and used, *after* the three less general generalizations have been developed and used.

Generalizing the Solution of a Problem

Among the most useful, and most difficult, generalizations is that known as *generalizing the solution of a problem*. To illustrate:

1. Tom can mow a lawn in 2 hours, and Joe can mow it in 3 hours. How long would it take both boys working together to mow the lawn?

The solution (as given by superior 14-year-olds) is somewhat as follows:

- (a) In 1 hour Tom can mow $\frac{1}{2}$ of the lawn.
- (b) In 1 hour Joe can mow $\frac{1}{3}$ of the lawn.
- (c) Working together for 1 hour, the boys can mow $\frac{1}{2} + \frac{1}{3}$, or $\frac{5}{6}$ of the lawn.
- (d) Since they mow $\frac{5}{6}$ of the lawn in 1 hour, they could mow all of it in $\frac{6}{5} \times 1$ hour, or $\frac{6}{5}$ hours.

The generalized solution takes the form:

- (a) Representing the time (number of hours) by a and b , we see that in 1 hour both boys can mow $\frac{1}{a} + \frac{1}{b}$ or $\frac{a+b}{ab}$ of the lawn.
- (b) Since they can mow $\frac{a+b}{ab}$ of it in 1 hour, they could mow all of it in $\frac{ab}{a+b} \times 1$ hour, or $\frac{ab}{a+b}$ hours.

The generalized solution provides a formula for solving all such problems:

$$T = \frac{ab}{a+b}.$$

2. Find the cost C of n candy bars which sell at a for b cents.

When the solution is not apparent, learners frequently convert the general problem into a particular problem, such as "Find the cost of 12 candy bars which sell at 2 for 5 cents." Having found the product of the number of 2's in 12, and 5 cents, they then can generalize the solution and obtain:

$$C = \frac{n}{a} \times b.$$

PRACTICE IN THINKING ABOUT GENERALIZATIONS

Teachers wisely employ classroom discussion of arithmetic statements that may be *always true*, *never true*, or *sometimes true*. For this purpose, such statements as the following are examined and classified.

1. The sum of two addends is larger than either addend.
2. Either of two addends equals their sum decreased by the other.
3. The minuend decreased by the subtrahend equals the remainder or difference.
4. Minuends are larger than subtrahends.
5. The subtrahend, plus the remainder, equals the minuend.
6. $a + 0 = 0$.
7. $a - 0 = 0$.
8. $0 - a = a$.
9. $a + a + a + a = 4 \times a$.
10. $a \times 0 = 0$.
11. If $a \times b = c$, then $a = \frac{c}{b}$ and $b = \frac{c}{a}$.
12. Multiplier times multiplicand equals product.
13. The multiplier and the multiplicand are factors of the product.
14. The product is larger than the multiplicand.
15. If the multiplicands are kept the same, and the multipliers become larger, then the products become smaller.
16. Dividend divided by divisor equals quotient.
17. The divisor and the quotient are factors of the dividend.

18. Dividend divided by quotient equals divisor.
19. A multiplication fact suggests a division fact.
20. A division fact suggests a multiplication fact.
21. Dividends are larger than divisors.
22. Remainders are smaller than divisors.
23. The sum of two odd numbers is an even number.
24. The difference between two odd numbers is an even number.
25. Multiples of odd numbers are odd numbers.

THE ROLE OF THE TEXTBOOK IN TEACHING PUPILS TO GENERALIZE

Are textbooks a help or a hindrance in guiding pupils to discover and generalize? Do not textbooks differ greatly in their provision for discovery and generalization?

Fortunately, the type of text based upon the psychology of giving the generalization or rule, followed by illustrative solutions, followed by exercises and problems employing the generalization are being supplanted by texts which challenge the learner to discover, to experiment, to reason, to find alternative methods, to be his own teacher, to evaluate, and then to practice for competence. Where the latter psychology is used, we must conclude that texts are an indispensable, if not the most productive, aid to the learner.

Texts are most effective when the teacher makes sure that the learner is prepared (ready) to read them. In this connection the writer has stated:

1. In order to be able or ready to read mathematics, most pupils require an oral, exploratory, experimental, developmental, discussion type of teaching, designed to build and enlarge the concepts and generalizations. During this period, the essential oral and written vocabulary becomes familiar and meaningful.

2. Following the above procedure, pupils having no special disabilities are able to read with comprehension statements and questions about the topics just developed. Emphasis is upon the recognition of ideas and the relationships among them.

3. Mathematical reading is highly specialized (see chapter 9). The person best equipped to guide the reader is the mathematics

teacher who conceives of the teaching of reading of mathematical ideas as an integral part of the learning in his field.

SUMMARY

We have suggested procedures for helping learners discover some of the great generalizations of arithmetic computation and problem solving. A variety of inductive and experimental procedures have been illustrated.

We have recognized the role of readiness for profitable attacks on the problem of discovering the generalization—possession of the prerequisite concepts and the necessary mental maturity.

From the beginning we have assumed that learners with the above prerequisites can and should generalize and look for relationships, thus making ever greater unity out of what would otherwise appear as discrete items of knowledge.

Generalizing is an active, meaningful, high-level type of mental behavior. Success in generalization pays the learner most generous dividends in satisfaction and in increased resourcefulness—in finding out for himself what he had never previously known. It is the *sine qua non* of creative thinking.

Arithmetic in Kindergarten and Grades 1 and 2

HERBERT F. SPITZER

THE PROBLEM OF what arithmetic to teach in kindergarten and grades 1 and 2 has long been a matter of some concern to those responsible for instruction in the elementary school. About 35 years ago there was much criticism because memorizing of number facts (often not very well understood by pupils) comprised the major part of instruction. Such criticisms resulted in elimination of nearly all arithmetic from some first grade programs and severely restricted teaching of the subject in the second grade. Then as meaning and understanding began to receive more emphasis in instructional programs, there was a gradual increase in the amount of arithmetic taught, especially in grades 1 and 2. There has, however, never been a widely accepted program of instruction for these three grades. As a result, there is great difficulty in producing instructional material for grade 3 that is neither too difficult for those who have had little instruction in the preceding grades nor too easy for those who have had such instruction.

Growing awareness of the importance of mathematics in our society and the close scrutiny that all mathematics programs are now undergoing have resulted in an increased concern about the arithmetic programs in kindergarten and grades 1 and 2. The fact that we are devoting a chapter to the topic is indicative of current concern.

The purpose of this report is to provide material and suggestions for study in this important division of arithmetic instruction. The report consists of three major parts. First, an attempt is made to determine current practices and to identify their major weaknesses. Second, a critical analysis is made of current instructional programs. This analysis and some important issues in arithmetic instruction are identified and discussed. Third, suggested guides for selection of content and an outline of content for arithmetic programs for kindergarten, grade 1, and grade 2 are offered.

ARITHMETIC CURRICULA AS PRESENTED IN PUBLISHED MATERIALS

Three major sources of information were used to determine current arithmetic practice. These were (a) courses of study, (b) professional books on the teaching of arithmetic and sections dealing with arithmetic in professional books on teaching in kindergarten and primary grades, and (c) published arithmetical materials for pupil use.

The Kindergarten Curricula

Since there is little in the way of actual arithmetical materials for kindergarten pupils, the investigation at that level had to be based on courses of study and professional books. An examination of representative city and state courses of study showed that identification of arithmetic content for kindergarten would be difficult because so little content is described. The typical course of study devotes space to such matters as the need for meaningful arithmetic instruction, the vocabulary of arithmetic, and a description of activities where numbers are used. These activities are primarily in the fields of science, social studies, and literature

and, as is proper in such situations, numbers are used because they are needed in carrying out the literature, the science, or the social studies activity. Among the most popular number activities described in courses of study are those relating to use of measures, as in cooking or in keeping a record of the weather. Pictures showing children using cups and quarts, reading the thermometer, weighing objects, and the like are used in so many recent courses of study that one gets the impression that such uses of measures must be a very important part of the arithmetic program.

Courses of study contained few specific recommendations as to whether or not direct instruction should be given in such areas of arithmetic as counting, reading numbers written as numerals, using numbers to identify amounts, and solving children's problem situations involving quantity. Since the activities described imply that the above-named areas of arithmetic are needed, it must be assumed that pupil learning comes primarily from the giving of needed information by the teacher or other pupils.

Professional books on kindergarten education are even less specific in identifying arithmetical content than are the published courses of study. In some of the books the word *arithmetic* does not even appear. However, there are usually some broad statements, even in such references, which indicate the possible use of some arithmetic in the kindergarten program. The following is representative (2:152): "No plan for a kindergarten year can be complete unless it recognizes the necessity for giving the children a good preparation for the school work which they will meet in the next year and in years to come." Since arithmetic is an important part of a child's future school work, the preceding statement might be used to support the inclusion of some arithmetic in the kindergarten program.

In the few statements regarding arithmetic found in professional books on the kindergarten, there are some which are difficult to interpret. The following is an example (1:211): "One way of eliminating waste effort is to postpone practice exercises until the child has an adequate understanding of the meaning and uses of number in life situations." Does this mean that all learning of counting, reading of numbers, and the like should be left to life situations with no use made of good instructional exercises for helping children acquire some facility in these areas?

Some statements are even more difficult to interpret. The following is an example (4:265): "There is no such thing as kindergarten or first-grade arithmetic—we teach children what they are ready for . . ." When examined critically, such statements are of no value to students of teaching. It should be recognized, however, that even if the statements are of no significance to students of education, they may have led many people to assume that arithmetic has no place in the kindergarten program.

It is from such statements as the preceding that the arithmetic content recommended by writers in kindergarten education is to be identified. Instead of attempting such a difficult task, it is probably better to say that while writers in kindergarten education are not opposed to numbers in the program, they are not inclined to recommend any definite number program.

First and Second Grade Curricula

The arithmetic portions of courses of study for grades 1 and 2 are similar to those for the kindergarten in their emphasis on the importance of teaching meaningfully and in the use of activities. In other respects there are marked differences; one of the most striking of these is the listing of specific arithmetical content to be taught in first and second grade courses of study.

Not all courses of study are specific in listing content, but it is not difficult to identify the arithmetic content. There are marked variations among various courses of study in the extent and number of topics listed. For example, some limit experiences with numbers in grade 1 to the numbers 1 to 10 or 20 while others suggest the use of hundreds and thousands.

Among the most frequently appearing additional topics are the following: activities involving use of number (for example, playing store), games involving number (spinner games, ring toss), exercises concerned with the meaning of numbers, counting activities (bouncing ball), use of special teaching aids (peg board, abacus), geometric forms (square, circle, rectangle), arithmetical terms, and the easy addition and subtraction facts (sums and minuends of 10 or less).

All the recent professional books on the teaching of arithmetic either directly advocate or imply the desirability of teaching

arithmetic in grades 1 and 2. The content advocated by the various books varies from very limited special material, as in *Children Discover Arithmetic* (5), to the consideration of many topics and materials. On the matter of grade placement, the professional books vary from descriptions of the content for each grade level to such indefinite statements as *the pupils should have experiences with these materials*. It is, therefore, very difficult to make a summary or general statement regarding a curriculum based on the recommendations found in such books. If a statement were prepared, the curriculum for grades 1 and 2 would, as far as content is concerned, be even more varied than a curriculum based on courses of study. Activities and projects involving number, special arithmetical teaching aids such as place value frames, verbal problems (presented orally), and special exercises for bringing out the meaning of numbers and processes would be emphasized. Included in such a curriculum would be a number of practices that are at present of only theoretical value.

The third source of information on the first grade arithmetic curriculum—that of pupils' textbooks and the accompanying teachers' editions of such books—is much better than the other two. There are many companies which sell arithmetic materials for use in the first grade. For all practical purposes these books become the course of study in the systems where they are used.

An examination of current books shows that the following areas comprise the bulk of the arithmetic content in grade 1: counting from 1 to 10 and then to 100; reading and writing of numbers, 1 to 10 and then to 100; the cardinal meaning of the numbers 1 to 10; the collection idea as applied to the numbers from 10 to 100; addition and subtraction of amounts (sums to 10 and minuends of 10 or less); recognition of most commonly-used coins; some of the relatively simple aspects of telling time; and some work with such measures as the quart, cup, and pound.

Since textbooks are a good representation of the course of study, the content listed and the methods of instruction indicated in teachers' editions should be examined critically. Five outstanding characteristics revealed through such an examination are presented below:

1. Great emphasis is put on the numbers 1 through 10, as

shown by the following facts about four widely used first grade books: pages dealing with numbers larger than 10 comprised 7, 15, 15, and 23 percent of the total; the first page on which a number larger than 10 appears is after 45, 52, 53, and 60 percent of the pages have been passed; the number 100 does not appear until 60, 75, 76, and 87 percent of the pages have been passed; and not one of the books presents as a part of the instructional material a number larger than 100.

2. For the development of the meaning of numbers and of the basic ideas in addition and subtraction, pictures and drawings (or the objects for which the pictures and drawings are substitutes) are used almost exclusively. To show the meaning of five, for example, various objects in groups of five are presented. Then exercises are given, requiring the pupil to draw five pictures, dots, or marks. In beginning work with addition, two groups of objects are pictured, each is identified, and then the total is determined. Following initial work with pictures, numeral presentation of ideas is rapidly developed; these ideas then become the second major means of presenting meanings and foundations.

3. A small step-by-step presentation plan is offered. In learning to write numbers, for example, there is a page of 1's, of 2's, of 3's, and so on. In a similar manner, when addition is introduced much attention is first given to facts with sums of 3 and 4, or 2, 3, 4, and 5, followed by a study of the facts with a sum of 6, and so on.

4. The settings used to introduce and to develop addition facts are of the showing or telling type. As indicated in 2 above, when addition is introduced the amount in each of two divisions of a set of pictured objects is determined and then the total amount is determined. Next the pupil is told that the total is the same as the two amounts; for example, that *one and three are four*.

5. Exercises designed for teaching the process of addition and for the study of addition of numbers written as numerals are introduced rather early in grade 1. In the four texts referred to above, the numeral form of the addition of numbers is introduced after 23, 34, 40, and 70 percent of the pages have been passed. The numeral form is then used almost exclusively in exercises designed for the study of addition.

Books for grade 2 stress the same major content as that listed

for grade 1, with less emphasis on counting and much more on addition and subtraction. The extent of this emphasis is shown by the fact that four representative second grade books give 60 percent or more of their space to addition and subtraction. In addition to the content listed for grade 1, most second grade books offer some verbal problems. In each of the various areas of arithmetic the work in grade 2 is more extensive than that for grade 1, but the amount of additional content covered is not extensive. For example, in most second grade books only the basic facts with sums through 12 or 14 and the addition of two-digit numbers without carrying are included. Since the sums through 10 are introduced in the first grade, it can be seen that not a great deal of new content is added in the second grade.

The reader should recognize that the statements in the preceding paragraphs refer to curricula representative of the average or most common practices. There are specific programs of instruction in use that are superior to the rather limited text-type program with its supplementary activities. Such superior programs are not common and no descriptions are available. They are often dependent upon skillful and highly motivated teachers, pupils of special ability, abundant and varied instructional materials, and the like. In addition, these superior programs are characterized by extensive use of orally-presented quantitative situations that are generally considered too difficult for pupils of the same grade level. Because of the features mentioned above, such superior programs will not be considered in the discussion that follows.

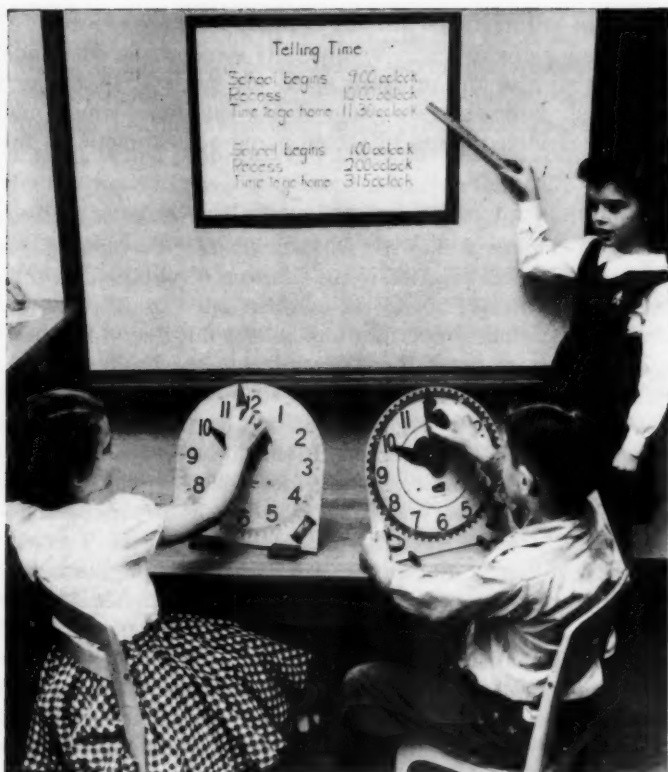
A CRITICAL ANALYSIS OF CURRENT CURRICULA IN KINDERGARTEN AND GRADES 1 AND 2

The picture of representative curricula presented in the two preceding sections is not entirely satisfactory to those who are anxious to improve the instruction of arithmetic in kindergarten and grades 1 and 2. First, it is practically impossible to identify from the published materials representative arithmetic programs in the kindergarten. The program at this level, if any, is very dependent upon the individual teacher and the teacher cannot find in the literature any clear-cut guides as to the nature or

amount of arithmetic to include. Second, there are aspects of the programs for grades 1 and 2, especially some of the outstanding characteristics of the program listed for grade 1, that may be questioned. Following are some of the questions which might well be considered issues in the teaching of arithmetic at these grade levels.

Should beginning instruction in the various areas of arithmetic emphasize only the first or very simple aspects of the topic? A good illustration of this practice is the limiting of pupils' first school experiences with number to the numbers one through ten. While admittedly the most important numbers by far, they would hardly have that importance if there were not others. The pupil held to the meager (might well be called a starvation diet) experiences with numbers one through ten will not be likely to grasp the important role of these numbers. It should be recognized that the great majority of pupils entering first grade already know how to count to ten and beyond. They already know some of the addition and subtraction facts involving these numbers. To confine such pupils' first grade number experiences to the numbers one through ten is not likely to generate much enthusiasm for the study of numbers. Even the school environment in which these first grade pupils live contains many numbers larger than ten—for example, the dates which are often written on the board and calendars which show when Halloween, Thanksgiving, Christmas, and pupils' birthdays come. Usually even the simplest picture books which the pupil uses have more than ten pages, and most pupils have house and telephone numbers larger than ten. The pupil is weighed and his height measured as a part of the school program. Here, too, numbers larger than ten are used. If the arithmetic course limits experience to numbers of ten or less, such number situations as the ones identified above and many others which are integral parts of good first grade programs do not become a part of the arithmetic program. The common arithmetic curriculum, then, may be a bar to good learning situations.

This criticism of the limitation to the simple aspects of content applies not only to counting, meaning of numbers, and reading and writing of numbers, but also to addition and subtraction. In



these areas there is little justification for spending much time in grade 1 teaching only the easy addition facts and then teaching the rest of the basic facts the next year. The extent of this emphasis on the simple in addition is shown by the fact that one widely used second grade book devotes six successive pages to adding one to the numbers one through nine. When such practices are followed, the pupil who already knows the facts either becomes bored or becomes lost in searching for some reason for doing unnecessary study.

Can it be assumed that first grade pupils will want to do arithmetic exercises just because teacher or text provides an assignment?

To illustrate, let's consider the typical text exercises in counting. Here is one: "Count the toy soldiers in the picture." This would, of course, be supplemented by a teacher directive, such as "Write the number that tells how many." The point questioned is this: What is there in such an exercise to give the pupil any notion as to why he is to count except to please the teacher? This procedure for counting exercises is in sharp contrast with the kind of exercises used in the best science, social studies, or reading programs. In those areas a setting is developed in which the pupil has a need for doing the suggested exercises. It seems that arithmetic instruction might also profit from the development of use or need settings. How to create such settings is one of the big tasks of instruction.

*Is the development of the basic ideas of fundamental processes best accomplished through almost exclusive use of pictures or even of the objects which the pictures represent? Consider, for example, the development of the first addition in the typical textbook program. Pictures are presented in which the combination of two distinct amounts may be inferred. Pupils are directed to determine how many in each amount and to determine the total. To illustrate for such a pictured situation as "●●● ●●●," pupils are led to say that in the first division of the picture there are three dots, four in the next, and seven in all. Then the statement "Three and four are seven" is made and is written in this manner: "3 and 4 are 7." We ask the crucial question: *Is such use of pictures the best way to help pupils grasp the basic idea of addition; namely, that when you add two amounts, you think an amount that is the equivalent of the two?* It seems as though the picturing of three objects and four objects does not show very well that these two amounts are equal to seven. This use of pictures in teaching the basic notions of addition in the manner indicated is, in the opinion of the writer, not as effective as it might be because the pupil has no use or need for showing or finding out that the two amounts (3 and 4) are equal to the total (7). This development of the process of addition would, it seems, be a much more effective learning situation if the pupils needed the pictures or objects for finding the answer, or demonstrating the correctness of an answer, to a problem. Furthermore, the finding of answers to addition*

word problems fosters thinking the aggregate of the two amounts. Pictures just show the two amounts, with no indication of the fact that an equivalent of the two is to be found. Above all other points on this matter of use of pictures and the like, it should be recognized that finding totals by counting pictures or by putting together two amounts representing the two numbers is not adding. The pupil who relies on objects and pictures to find answers has made little progress, arithmetically speaking. Therefore, a program that gives a pupil experience only with objects, pictures, and drawings is not only insufficient, but it may also deprive pupils of the opportunity to do the kind of thinking that is essential to arithmetic.

Is an explanation of an arithmetic fact or process the best way to introduce such facts or processes in the study of arithmetic? It is generally accepted that the learner is most likely to get the significance of an explanation if he has some experience with the situation before an explanation is offered. It would seem, then, that the first experience with a new process should be in a situation where the pupil can see the necessity for such a process. The telling, showing, explaining type of introduction to facts that is typical of current practice in grades 1 and 2 may be questionable.

Are the early presentation of and the heavy reliance on the numeral form of addition exercises (for example, $\begin{array}{r} 4 \\ + 2 \\ \hline \end{array}$, $4 + 2 = \text{—}$, 4 and 2 are —) good learning procedure? Since this numeral form of presenting addition questions is very different from the other question situations with which the first grade child has experience, it undoubtedly creates a special learning difficulty for some pupils. When confronted with "4 and 2 are —," some pupils are not likely to see this as a different way of asking the familiar type of question, "When I put four with two, how many will there be?" It would seem that the use of several ways of asking addition questions, and most certainly the use of familiar questions, would make for more effective learning. Furthermore, it is generally agreed that it was the early and exclusive use of the numeral form of basic facts in primary grade arithmetic study that led

to the meaningless, verbalistic type of learning which exponents of the so-called meaning theory attacked with such vigor. The extensive use of pictures prior to the introduction of numerals and the use of *and* and *are* instead of *plus* and *equals* does not completely eliminate the possibility of a type of learning that is not very meaningful. This use of *and* for *plus* and *are* for *equals*, while eliminating two words that are not very familiar to primary children, introduces two very troublesome matters. It has resulted in the use of a plural verb *are* for the singular mathematical word *equals* and has also resulted in a statement that is logically incorrect; namely, that four and three are seven. Four and three are two distinct numbers that are equal to seven but they are not seven.

Why do arithmetic programs in grades 1 and 2 not make extensive use of word problems? The word problem is a much more effective way of presenting quantitative situations to children than is a dot drawing accompanied by an addition or subtraction fact presented with numerals. Situations closely resembling word problems occur frequently in the pupils' out-of-school experiences. For example, a child may have three pennies. He knows he does not have enough to buy a five-cent candy bar. He wants to know how many more pennies he needs. Such situations can be used to provide the setting for use of actual things or for use of representative things such as dots or marks to show that two more must be put with three to equal five. In the opinion of the writer, the addition word problem is the place to start a child's thinking in addition. An attempted pictured representation of an addition is not the place to start. Furthermore, writing in arithmetic in the early stages is a record of pupils' thinking, not the steps to thinking as might be inferred by looking at children's books in arithmetic. The avoidance of problems is in part due to the mistaken notion that pupils should read the problems, and, of course, the reading ability of the average first grade child is not sufficiently developed to enable him to do this. Pupils' inability to read is not, however, justification for omitting problems from the program. The teacher can present these problem situations orally to the pupils.

Why should the few problems that are presented in first and second grade books deal only with very simple addition and subtraction situations? As we have stated, the word problem is one of the best ways of providing number foundation experiences. Surely the areas of multiplication and division are as much in need of foundation experiences as are the areas of addition and subtraction. Furthermore, the counting of five more to a larger number (18, for example) to get the total may be a better way of building a foundation for addition than is exclusive attention on such simple situations as $5 + 2$ and $3 + 1$.

Why do pupils of superior ability seldom show much enthusiasm for the arithmetic of school? The most plausible answer to this question is the fact that the arithmetic content presented for study is too meager to challenge, has probably already been mastered, and therefore does not give the pupils an opportunity to do something that will bring a feeling of accomplishment. It is in the best interests of these superior pupils and of society that the early work in arithmetic be interesting and challenging. To accomplish this requires something more than the current arithmetic programs.

If the picture of arithmetic instruction in kindergarten and first and second grades presented in the preceding section is correct, or even partially correct, there is a need for extensive and careful study of the situation. Such a study should, of course, probe more deeply into the program than was done in the preparation of the present report, and should examine carefully the premises or foundations of arithmetical practices. For purposes of this report only three major sources are identified: (a) the field of mathematics, (b) the principles and practices of primary grade teaching and (c) the everyday uses of arithmetic.

The use of such divergent sources has produced some of the problems that now exist in arithmetic instructional programs. For example, consideration of the field of mathematics has led to the attempt to include material on the decimal nature of our number system and has led to the development of procedures for teaching the nature of our notational scheme. To show either of these aspects of our system in a way that makes sense would

require, in addition to the units or ones numbers, the use of at least tens and hundreds, and perhaps even thousands numbers. To use other than ones numbers is, however, in conflict with one of the dogmas of teaching primary grade children; namely, that only the simplest concepts should be used in beginning instruction, and these simple concepts should be learned well before the pupil is confronted with the next step.

From the illustration cited, it may be safely concluded that conflicts are going to occur if several sources are used for the basic foundations of an arithmetic program.

Can an adequate beginning arithmetic program for elementary schools be based on one source and thus avoid the conflict of interests due to use of several sources? For example, can an adequate program be based entirely on the tenets of mathematics? The answer to such questions is a definite *no*. A number of such proposals have been tried and some are being tried now. Two programs being used in whole or in part in some schools have a single base which is primarily mathematical. Reference is here made to the system of structural arithmetic developed by Stern (5) and to the system of arithmetic of colors developed by Cuisenaire and promoted by Gattegno (3).

None of the past proposals of this type have been satisfactory and it is doubtful whether the current proposals will last. They may work well for some pupils, especially the mathematically inclined, but elementary schools and instruction in them are for all children and not just for the few who have special aptitude or interest.

Can we start with a mathematical base and incorporate into a program the best principles and practices of elementary education? It would appear that *yes* is the answer to that question, but in the last 20 years most of those concerned with producing materials and suggestions for the arithmetic program have been trying to do just what the question suggests. The answer, then, is that so far we have not been able to develop a program free of inconsistencies brought about by use of a base built on divergent sources, but it is believed that progress has been made.

The preceding discussion is a result of taking a look at the sources on which primary grade arithmetic is based. Another area for examination in a critical study of the current arithmetic situation would require a close look at actual teaching practices. The space allotted to this report does not permit the inclusion of questions and issues which might arise from such a look.

SUGGESTIONS FOR THE PROGRAM OF INSTRUCTION

The Kindergarten Program

In the absence of adequate descriptions of the best kindergarten programs, the writer bases his suggestions on what he considers to be representative of good kindergarten programs. Such programs provide experiences which often require the identification of quantities. For example, pupils engage in activities using such definite statements of quantity as "You need two more blocks to finish," "The grasshopper has six legs," and "The puppy has five toes." Good kindergarten programs give pupils an opportunity to plant seeds, to grow plants, to have animals for observation, and the like. Where such activities are used effectively, records are kept of the time (in days) required for seeds to come up and of the amount of food needed for the animals in terms of number of stalks of green feed or number of pieces of prepared food. Some of the literature read to kindergarten children in these good programs contains numerical material. If pupils are to grasp the import of numerical statements, number knowledge is needed. To help pupils acquire this knowledge, the teacher may show the actual quantity, count the separate parts, draw marks to show the number, give the number of sounds (for example, the five peals of the bell), and the like.

Other instances could be cited to indicate how numbers are needed in kindergarten programs, but the preceding should be sufficient to show that if pupils are to have truly rich experiences in kindergarten, some number knowledge is essential.

The number program suggested for kindergarten is similar to the type described above. It affords children number experiences as an integral part of the regular kindergarten program. The

numbers instruction consists of direct and specific attempts by the teacher to help pupils get an idea of the numbers involved in day-by-day activities of kindergarten.

While specific kindergarten number experiences will vary from program to program—depending upon the projects under way, the literature used, the science and social studies experiences, and the like—there will be a general core of number experiences. This will be, for the most part, from the areas of arithmetic presented in the list below.

1. Counting used to identify which one. This kind of counting is used in the dramatization of such stories as "The Six Little Squirrels."

2. Counting to find how many. This kind of counting is used in all situations where the amount must be expressed with numbers. The following are examples: "You may have three pieces," and "It has been five days since we planted the seeds."

3. Counting as used in rhymes, jingles, finger plays and special counting activities. Examples of these uses of number are "1, 2, 3, 4, 5, I caught a hare alive," "One finger, one thumb keep moving," and "Simon says, 'Turn around 3 times.'"

4. One-to-one correspondence as used in supplying each member of a group with a sheet of paper, in putting together jigsaw puzzles, and similar situations where matching of quantities is required.

5. Use of number names to identify or to describe. The following are examples: "At 10 o'clock our radio program begins," "I see five kittens in the picture," and "He paid five dollars for his new shoes."

6. Use of indefinite quantitative terms, such as *big* (including *bigger* and *biggest*), *long*, *short*, *small*, *fast*, *slow*, *few*, *many*, and *little* in the everyday expressions of the classroom. The following is an illustration: "Now we need a bigger person to play the part of the big bear. Who can it be?"

The illustrative statements in each of the six areas listed show arithmetic in a use or need situation. This use or need for numbers in the regular kindergarten activities is the important point to note; the numbers program in kindergarten should be confined to numbers that are used or needed in the regular room activities.

This means that neither mathematics nor the elementary education principles will play a major role in the determination of arithmetic content. It also means that kindergarten number programs, just as now, will vary from school system to school system. Thus it may appear that the suggested program is not an improvement. Careful consideration will show, however, that the recommended program is definitely superior to present practice. The superiority is due to the suggestion that where numbers are used in literature, science, social studies, and everyday activities, pupils are to be helped in getting the import of the numbers used. Numbers, when they occur, are not then to be avoided or neglected. Furthermore, if such a program is recommended to teachers, they will not feel uneasy about teaching pupils something about numbers in activities where numbers are essential. This would be a marked improvement over the present situation.

Basing the kindergarten program entirely on the uses of number in regular classroom activities has an important implication other than selection of content. If instruction in number is to help pupils get the import of numbers as used in daily activities, in literature, and the like, then there is no need for pupils to master any of the various aspects of number. Back of our desire to have pupils master arithmetical content is the conviction that the knowledge will be needed later in connection with similar quantitative situations. In the suggested kindergarten program the child has an immediate need for the number knowledge with which instruction is concerned; the use of such number knowledge in the future is not a factor. Of course it is recognized that such instruction will give pupils knowledge for future use, but the point to remember is that instruction in number in the kindergarten is for *now*, the *immediate*, and not for the future. There is, then, no need in kindergarten to teach any aspect of number for mastery.

The First Grade Program

In first grade the teaching of some arithmetic is an established practice; suggestions for a program of instruction at this level have a better basis, therefore, than those for the kindergarten level.

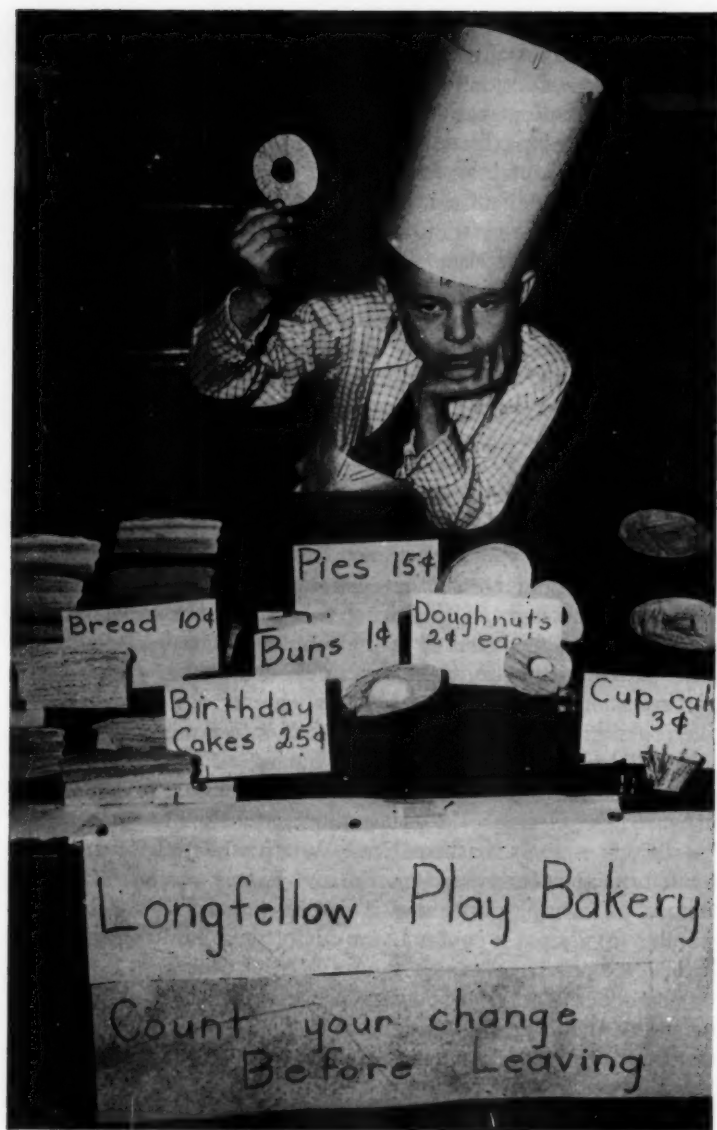
The chief objectives of the suggested program that follows are (a) to provide a set of guides or principles to be used in the selection of content and instructional procedures, and (b) to provide an outline of content based on the proposed guides. In the preparation of guides the three major factors (mathematics, principles of teaching in elementary school, and use) which have been the sources of arithmetical content in the past have been assigned important roles.

It is suggested that the field of arithmetic or the broader field of mathematics be given first consideration in the selection of the content for the first grade arithmetic program. Specifically, this means that content for possible use in the program be chosen first from the field of arithmetic. In this selection prime consideration should be given to the logical first steps or so-called foundation aspects of the subject. These are the simplest and the easiest aspects, and thus the second basic source in the selection of content (the principles of elementary education) has been used.

The third major source of content, the use principle, will be employed in the following manner. Unless there is a logical pupil use or need for content, that content will be eliminated. For example, counting (just knowing the order of number names) would certainly be included in the simplest content selected from the field of arithmetic, but a use or need that first grade pupils can see should be presented before such counting is included in the program. The use principle then has veto power over content and will certainly exert major influence on the instructional procedures used.

Stated as questions, the guides for the selection of content are: (a) Is this content mathematically worth while; that is, does it contribute to learning arithmetic? (b) Is this content of such a nature that simple aspects of it can be presented to first grade pupils? (c) Can this content be initially presented to first grade pupils in a setting where they see a need or use?

The identification of content is one part of an arithmetic program; the other important part is the instructional procedure used in presenting this content. Guides or principles for instructional procedures, if they can be specific, are just as important as are guides for the selection of content. Unfortunately, the



desired specificity in this area is hardly attainable, and the statements that follow are offered with some misgivings.

1. Instruction in the various areas of arithmetic in first grade should be initiated with situations in which pupils can see a use or need for the fact or procedure being taught. When such conditions exist the extent of first instruction will be governed almost wholly by this use or need. If, for example, a counting rhyme situation is used to identify *the one*, or *it*, then instructional emphasis will be on learning the order of number names and accompanying words in order to find *the one*.

2. Instruction in every area of arithmetic presented should go far enough to give pupils an opportunity to see the system or to grasp its significance. For example, counting should extend through the tens and hundreds so that pupils may have an opportunity to see that the order of the numbers from 1 to 10 is applicable again and again.

3. The words *opportunity to see* used twice in the preceding paragraph point to another guide to instruction. The content of arithmetic in the first grade should be offered in this spirit of opportunity to learn and not content to be mastered. The numbers program in first grade is thought of as a foundation program. Since no one knows what the foundation number experiences are, we can hardly insist that a pupil master this concept or this procedure in order to continue his study. The various areas of arithmetic presented in first grade, while all-important in the field, do not require pupil mastery at this time.

4. In helping pupils acquire sound ideas, it is often helpful to have actual objects to show and manipulate, to have or make drawings representing quantity, or to have such diagrammatic representation of quantity as the number line. There are two important points to note in connection with the use of such materials: first, they are used for a purpose—to clarify a question or to show the meaning of a statement; second, these special visual materials are only the basis for more advanced thinking. Unless instruction permits pupils to go beyond the visual materials, nothing of any consequence has been accomplished.

5. Because the verbal problem provides pupils with a good setting for acquiring an understanding of many quantitative

situations, such problems presented orally by the teacher are an important part of the instructional program.

While many other guides to instruction might be offered, the rather nebulous character of most such guides warrants limiting the list to the five presented above. Perhaps the most important point about the guides is the fact that even though the content of the numbers program is important arithmetic, instruction in grade 1 is directed only toward providing pupils with experiences that may result in the acquisition of knowledge that is immediately useful and may become the basis or foundation for later study of the subject. There is no specific list of number facts to be mastered. Another point about the guides which deserves second mention is the fact that all content is to be introduced in a situation where first grade pupils can see a use or need for that which is being taught.

Seven major divisions of the content of the suggested first grade arithmetic program are presented below. A brief statement calling attention to some important aspect of that division is included with each.

Outline of a Suggested First Grade Program

1. *One-to-One Correspondence.* The simplest means of showing the equality (or non-equality) of two quantities is by matching the parts. This unsophisticated, immature way of dealing with quantities is essentially the basis for counting and therefore the basis of all the fundamental operations. Although one-to-one correspondence might well be considered a pre-counting experience, this can hardly be used for justifying its inclusion in the program, since practically all pupils can count when they enter first grade. Its role here is chiefly in proof or in showing that quantitative statements are true. For example, to show that the equality established by counting groups—one of three, another of four, and a third of seven—is correct, the parts of the two small groups are matched with the parts of the large group. This simple, though cumbersome, means is used to show that a shorter (counting) means is true. One-to-one correspondence has a similar important role to play in helping pupils to understand the fundamental processes of

addition, subtraction, multiplication, and division. This part of the first grade program, then, consists of more than a few exercises in which pupils see if there are enough bonnets for the dolls pictured.

2. *Counting to 1000.* Since most first grade pupils know how to count when they enter the grade, the school program is designed to extend their ability in this area and to provide a review of beginning counting which brings out its uses and major characteristics. The major characteristics emphasized are the order of the ones whether in tens or hundreds and the relationship in names between the ones, tens, and hundreds.
3. *Meanings of Numbers to 1000.* Major emphasis is on the cardinal idea, with the base idea used to help pupils get a more easily comprehended notion of tens and hundreds numbers. Many special instructional aids, such as meaning of numbers charts, dot drawings, objects for pupil and teacher manipulation, number lines, and place value charts, are used.
4. *Reading and Writing of Numbers, Including Hundreds.* Such common need situations as reading and recording dates, number of pupils present, the height and weight of pupils, and the like provide settings for the introduction and continued use of this phase of arithmetic. To help pupils get better ideas of the meaning of numbers, different ways of expressing quantity are used. For example, pupils are told that thirty is the equivalent of three tens and that the amount may be written 30 or 10, 10, and 10. In a similar manner 100, 100, and 40 may be used for two hundred forty.
5. *Fractions and Geometric Forms.* This area of arithmetic is restricted to those common fractions ($\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{2}{3}$, $\frac{3}{4}$) and those geometric forms (circles, squares, triangles, square corner, straight line) which occur frequently in the environment of first grade children.
6. *Measures and Weights.* The program in this area consists of experiences with such measures as the inch and foot; the pound and ounce; minute, hour, week, and month; cups, pints, quarts, and gallon; and all the coins.
7. *Orally Presented Quantitative Situations.* Illustrations of situations used in this area of arithmetic are the following.

- (a) Jane's birthday is on October 21. Fred's is on October 19. Whose birthday comes first?
- (b) Robert is writing the numbers 10 to 20. He has just written 11, 12, and 13. What are the next two numbers he should write?
- (c) Tim and John needed 8 wheels to make toy wagons. John found 3 wheels and Tim found 5. Did they find enough wheels?
- (d) Bob gave away 2 of his 6 pieces of paper. How many pieces did he have left?
- (e) On each side of a piece of paper Mary drew three pictures. How many pictures did she draw on the paper?
- (f) Tom had 4 cookies that he wanted to share with his friend Joe. How many cookies should he give to Joe? How many cookies should he keep for himself?

One teacher initiated consideration of the first exercises above with the following: "I am going to make a statement and then ask you a question about the statement. Listen carefully. Here is the statement. . . ." After a brief pause, several different pupils were given an opportunity to respond, and then by using the calendar it was shown that the 19th comes before the 21st.

For all such situations pupils are first given an opportunity to think the answer. Then objects, drawings, or such devices as the number line are used to find the answer if it has not been given, or to show that the answer is correct, or to show how the answer might be found. Of course this finding of the answer is a group project with the teacher doing the necessary drawing and directing the thinking of the pupils. This method of showing how answers can be found, plus the problem setting, provides valuable number experiences for pupils.

Special attention should be given to the fact that problems representing all four processes (addition, subtraction, multiplication, and division) are used. In situations using multiplication and division, solutions will seldom involve the actual processes of multiplication and division, and showing that the answer is correct will always involve the use of objects, drawings, and such devices as the number line. The problems part of the program is

considered one of the most important of the pupil's foundation number experiences.

Another part of every arithmetic program (not included above because it is not content) is an inventory or means of acquiring information on the pupils' arithmetical achievement at the beginning of the instructional period. The inventory test checks the pupil's ability to count and read numbers, his grasp of the meaning of whole numbers and fractions, his ability to solve problems (presented orally), his knowledge of common measures and coins, and his knowledge of simple geometric forms.

The most obvious differences between the suggested program above and present programs are the inclusion here of numbers to 1000 and orally presented quantitative situations, and the omission here of common addition and subtraction exercises. The inclusion of the hundreds numbers in counting, in meaning of numbers, and in reading and writing of numbers is based on both the use or need guide and the math guide. Pupils have a need for some hundreds numbers (for example, house numbers, page numbers, and other numbers of everyday affairs), and notation cannot be taught well unless numbers larger than ten are used. Furthermore, the significance of the small numbers is more likely to be grasped if 10's and 100's numbers are considered. This, then, is not an attempt to crowd more arithmetic into the first grade program, but is a deliberate plan to include enough to make its teaching meaningful.

The orally presented quantitative situations are included in the list because it is believed that such word descriptions and questions involving numbers provide the best setting for acquisition of basic ideas of fundamental processes in arithmetic. In addition, these word problems make it possible for a pupil to see sense in the use of drawings or objects. If, for example, a pupil's answer to a word problem involving the adding of 4 and 2 is challenged, he has to use marks or objects to show that the sum is 6. The word problem provides a setting for pupils to do the sort of thinking that is needed in adding, subtracting, multiplying, and dividing. Oral presentation circumvents pupil difficulties with reading arithmetical algorithms. It is believed that extensive work with orally presented arithmetic word problems will result

in the learning of some facts and will provide a good foundation for the study for mastery of the basic facts in the familiar arith-

metical form—for example, $\frac{4}{7} + 3$. Furthermore, this use of oral

word problems will eliminate some of the possibilities of too early study-for-mastery exercises.

The Second Grade Program

The program suggested for grade 2 includes the seven major divisions recommended for grade 1 plus systematic instruction for beginning to master the processes of addition and subtraction. Second grade instruction for the seven areas already studied in grade 1 is introduced with a new use-setting for two reasons: first, the able pupil will find something of interest; second, in such situations the teacher can observe deficiencies without putting pupils in a formal test situation. Since there are nearly always some pupils who are deficient, there is a need for going back to simpler aspects of the topic. This work will certainly be similar to that recommended for grade 1. It is important that a need for this review or for re-use of old material be evident to the pupils.

In addition to the review of the seven major divisions of the first grade program, more difficult aspects of each division will be introduced.

This review of and addition to the program of grade 1 comprise the major part of the arithmetic program of the first semester of the second grade. In this semester the work, like that recommended for grade 1, is concerned primarily with providing good background experiences. There is no attempt at mastery.

In the second semester of the second grade a new phase of instruction, that of mastering facts and processes, will be introduced. The processes of addition and subtraction and the basic facts of these two processes are recommended for this special study. Many pupils enter first grade knowing some addition and subtraction facts; they then spend a year and a half in a foundation program which includes much adding and subtracting for

the solution of problems. Hence there will be many second grade pupils who know many of the facts and understand the processes fairly well. To enable them, as well as those pupils who do not know facts, to see the reason for study, evidence is used regarding such factors as the time required to answer basic addition questions to show the need for knowing and responding with reasonable speed. Pupils frequently use matching or other immature procedures to show that answers are correct. Such procedures not only provide the beginner with a good basis for understanding, but also give the pupil who already knows the answer a better grasp of the fact and the process.

To give pupils a view of the addition process and to provide a sizable amount of content, the addition work should include the 81 basic facts (excluding zero) and addition of two- and three-digit numbers without carrying. The part of subtraction corresponding to the addition identified above would be included. This systematic work in addition and subtraction would comprise the bulk of the content in the second semester of second grade. It is recommended that the foundation work of the first semester, especially that concerned with counting, the meaning of numbers, reading and writing of numbers, and problems, be continued in the second semester, but with a greatly reduced time allotment.

The introduction stated that the purpose of this report is to provide materials and suggestions for the study of arithmetic instruction in kindergarten and grades 1 and 2. The reader will recognize that there are conditions, issues, and instructional program suggestions for study other than the ones included here. It is hoped that the report will stimulate those interested in arithmetic instruction in the early grades to consider critically the conditions and issues involved and, in the light of these conditions and issues, the suggested number experiences as well as other number experiences that may occur to the reader.

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Individual Differences

R. STEWART JONES and ROBERT E. PINGRY

THAT CHILDREN of the same age differ widely in the mental, physical and personal traits which affect schooling is now common knowledge (2) (40). Educators and psychologists have portrayed so well the diversity among children and have so emphasized the concomitant problems of teaching that many teachers have become convinced that adequate handling of so many different children in one class is impossible. Furthermore, the widespread attention to this topic has often carried with it the unfortunate implication that differences among pupils can and should somehow be *ironed out*. Nothing could be further from the truth. Differences among children will increase as a result of development and the school should have a hand in this development by making children more different rather than less. The problem, then, is not that individual differences exist, but what can be done in teaching to take account of these differences.

With nearly 100 percent of elementary age children in school, with rapidly increased birth rates since the last war, and with the holding power of the high school increasing each year (1) teachers

should anticipate not only more pupils but also greater diversity among them than was the case even a decade ago. Children are coming to school younger than ever before, they are staying in school longer than ever before, and they are remaining in classes with their chronological age mates more than in the past (24). Furthermore, there are far too few special classes and remedial staffs to handle even the extreme deviates (1). All of the foregoing facts do not point to a new problem; they focus sharply upon an old problem which few schools have tackled with sufficient imagination and vigor (15).

GENERAL PSYCHOLOGICAL CONSIDERATIONS

The number of human traits is unknown, but undoubtedly there are thousands. Cattell (6) has attempted to differentiate between the innumerable *surface* traits, which are apparent in behavior and personality, and *source* traits, which give rise to manifest behavior. Similarly, Thurstone (37) has developed methods for identifying the separate *primary mental abilities*. Both psychologists have attempted to group manageable and meaningful traits to describe individuals and their differences. Nevertheless, a composite description of a child, even using the factors of personality and ability, presents a complex picture. Every child is unique. But children of a given culture also have much in common. Cattell (6) attributes many of our similar traits to what he calls "environmental mold unities" and Gesell (14) has emphasized the many similarities in the sequential and orderly progression of development. If each child is seen as completely unique, it follows that instruction must be completely individualized; if all children are viewed as alike, teaching should be entirely a group process. Neither position is tenable, and all good teaching accepts a compromise somewhere between these extremes. Many traits can be measured. Children can be identified as possessing more or less of a characteristic that affects schooling, and the general characteristics of a class can be described. Clearly such measurement and the description that follows are only approximations, but they are extremely useful. Measurement has provided the knowledge that we have about differences among children and has given clues as to their causes.

The following propositions about individual differences should be a starting point for devising efficient educational procedures to handle them:

1. Children in any classroom, even in those where ability grouping is used, will differ widely in both general and special abilities and aptitudes (3).

2. Differences in both ability and achievement increase with age, the span or range of general ability increasing almost two-fold from the first to the eighth grade (9).

3. The various special abilities (for example, numerical fluency) correlate differentially with each other and with any measure of general ability that is presently in use (26).

4. Any school subject requires a variety of abilities (26). Competency in arithmetic, for example, calls for numerical fluency, numerical comprehension, conceptual ability, visual and auditory memory, etc.

5. Differences between the sexes in abilities and skills are slight, with some apparent advantage for the earlier maturing girl, especially in language skills, and some slight advantage for boys in mathematics (2).

6. Differences within an individual, that is, the variability of traits within a single child, are more than half as great as differences among children (9).

7. Although various abilities overlap, that is, intercorrelate, they are sufficiently independent to merit separate measurement (26).

8. Differences not accounted for by mental age measures are as important in determining school success as intellectual ones. One-fourth of retarded readers, for example, are above average in measured intelligence (11).

9. The effects of schooling that fails to consider differences among children seem to be cumulative. Children fall further and further away from the attainments of which they are capable at all levels of ability, but particularly at the higher ability levels (1).

10. Differences are not static. Children vary when they enter school and they develop at different rates. Consequently, diagnosis must be a continuous process.

The above generalizations will later be related to the problems of teaching and of school organization.

DIFFERENCES AMONG CHILDREN IN ARITHMETICAL SKILLS

Theoretically, differences in arithmetic achievement might be expected to vary as greatly as intelligence or as aptitude for arithmetic varies. Actually the variation is probably somewhat greater. The correlation between intelligence (or any of its factors or combinations thereof) and arithmetic achievement rarely exceeds .60 and in most cases proves to be lower. Obviously, unmeasured factors such as experience, emotional stability, modes of thinking, and attitudes contribute to success in arithmetic. The extent of the variability is portrayed in Table 1. As may clearly be seen

TABLE 1

Grade Level Scores in Arithmetic Earned by 519 Eighth-Grade Pupils

Grade level to which score entitled pupil	13	12	11	10	9	8	7	6	5	4
Number of children at each level	11	14	60	142	134	81	47	21	8	1

SOURCE: Adapted from (1).

there, many children are far beyond the competence that might erroneously be thought of as the standard for eighth-grade work. Witness the 25 children who were already as advanced as high school seniors or college freshmen. Similar findings by Frandsen (13) in five sixth-grade classes in Logan, Utah, showed 47 of the 195 children tested to be at a grade-equivalent at least one year ahead of norms. Such evidence dramatically portrays the problem of trying to teach the same arithmetic lesson to every pupil in a given class. However, on a scale of total achievement one gets neither sufficient diagnostic information to take specific action nor a clear picture of what the differences mean in operational terms. More relevant information on the meaning of such differences has been cited by Brownell (4), who tested 487 children in 20 fifth-grade classes in the component skills required for performing the operation of division. All these children were about to embark on this relatively complex arithmetical task. The results not only gave evidence of the expected wide range of achievement (half were not ready to begin this study) but also, and more important, identified the specific areas of weakness of each child.

Little can be gained at this point by further elaboration of the range of achievement in arithmetic. Any school that keeps records and uses them effectively has ample evidence of its own. Suffice to say here that such differences are not limited to the lower grades nor are they limited alone to computational skills. At every age level there is great variability in all areas of arithmetical and mathematical skill and understanding.

More important for this discussion is the fact that in the upper grades, in high school and college, and in adult life there are large numbers of people who do not possess even the minimum essentials of arithmetical skill, basic mathematical concepts, and related applications to everyday experience. The writer recently asked 124 graduate students what the diameter of the moon is. The answers ranged from one mile to 10 billion miles. Similar evidence of inability to use mathematical concepts is given by Horn (19), who cites a half-dozen studies to support this proposition. An anecdote in this regard is furnished by Richter (31), who describes the case of a teacher who asked a fourth-grade class to find the product of .08 and $\frac{1}{2}$. The teacher was berated by the father of one of the children; he had given up trying to solve his daughter's problem when told by a successful business acquaintance and a Spanish professor that the problem was impossible.

The school's goal, then, is not to reduce differences among children, but to take account of them in teaching and to attack the problem of widespread ignorance presently existing in this area. To this problem we now turn our attention.

FACTORS THAT CONTRIBUTE TO INDIVIDUAL DIFFERENCES AND TO EDUCATIONAL CASUALTIES IN ARITHMETIC

The factors that produce variability among pupils and that in some cases result in educational casualties include a variety of both innate characteristics and environmentally-produced traits. For purposes of the present discussion these factors are divided into three groups. Group I includes those basic factors over which the school has little control; Group II embraces those conditions arising outside the school that produce differences, but over which the school may have an ameliorating influence; and Group III contains those causes of difference that arise in the school itself.

Group I Factors

The school has practically no control over, but must make adequate teaching provisions for these factors, which include innate potential, accidents of birth and environment, and irreversible organic changes. Mental potential, one of the most important as a determiner of learning, is portrayed in Table 2.

As may be seen in Table 2, four-years-olds would be expected to vary from two to six years in mental age; at eight they would vary from four to twelve years; and at fourteen, from eight to twenty in mental age, a twelve-year span.

About 15 percent of all children in public schools have IQs between 75 and 90, and are sometimes referred to as the "dull-normal" group. These children should not be thought of as mentally handicapped, nor can one find much evidence to argue for separate classes for them. Because of the already noted intra-individual variability, some of these children will be at or above average in some of the components of arithmetical skill. As a group, however, they do constitute a serious problem for the school which uses an inflexible grade placement of topics in arithmetic or in any other subject. Another 4 to 5 percent of the school-age group may be classified as mentally handicapped; that is, with IQs between 50 and 80. In one of our most progressive states only 16 percent of this group is cared for in special classes. Undoubtedly, throughout the country as a whole, most of this group is in the regular classroom. Children whose measured IQs are below 50 are generally referred to as *trainable* rather than *educable*. Actually, however, many of these children are also to be found in the regular classroom.

At the other extreme, 1 to 5 percent of the children (depending on how defined) may be thought of as gifted or talented. In many cases the educational attainments of these children are far below the capacity of which they are capable (1).

Intra-individual variability, as already mentioned, further complicates the problem of caring for individual differences. In one study 25 children, all with IQs of 106, varied in mental age on different tests from one to almost eight years in such areas as memory, space perception, and verbal ability (38). In view of this type of diversity, Traxler (39) has emphasized the need for testing

TABLE 2
*Theoretical Distribution of Mental Ages for Groups of 100 Children
 with Given Chronological Ages*

(Based on a theoretical standard deviation of IQ of 16.6)

Mental Age	Chronological Age										
	4	5	6	7	8	9	10	11	12	13	14
22											
21											
20											1
19										1	2
18										1	4
17									1	4	8
16								1	3	7	12
15								2	7	13	15
14							2	6	12	16	16
13						1	5	12	17	17	15
12					1	4	12	18	19	16	12
11					3	11	19	21	17	13	8
10				2	11	21	23	18	12	7	4
9			1	9	22	25	19	12	7	4	2
8			7	23	28	21	12	6	3	1	1
7		5	24	31	22	11	5	2	1	1	
6	2	24	36	23	11	4	2	1			
5	23	41	24	9	3	1					
4	50	24	7	2	1						
3	23	5	1								
2	2										
Totals. . . .	100	99	100	99	102	99	99	99	99	101	100

SOURCE: Bibliography item (36).

that yields a profile of abilities. He suggests such tests as the Primary Mental Abilities Test, the Differential Aptitudes Test, and the Yale Educational Aptitudes Test as measures that will provide useful information for educational planning.

Beside mental abilities there are other conditions, such as physical handicaps and defective vision and hearing, that affect schooling. Only a small percentage of this latter group receives adequate special services, and consequently these children become a problem for the regular classroom teacher.

For all the Group I factors the school can do little to effect a change. It is futile to believe that even the most excellent teaching can create readiness for learning among those children whose mental potential is far below that of their classmates. The only recourse is to make appropriate adjustments in both curriculum and teaching methods. Suggestions for ways to handle these innate or uncontrollable differences will be discussed shortly.

Group II Factors

These factors arise outside the school, but are products of learning and maladaptation and are amenable to treatment within the school. Included are unfavorable attitudes, emotional disabilities, fears and anxieties, motivational differences and erroneous modes of thinking. These factors may be general personality characteristics, such as anxiety, which interfere with all school work, or they may be specific attitudes about arithmetic which arise in the home or community. One researcher found that children who are superior readers dislike mathematics more often than their peers (32). It is well known that girls, who are more proficient than boys in language skills, have less interest and ability in arithmetic. Girls rank arithmetic or mathematics in about seventh place when asked what school subjects are most valuable, while boys characteristically place mathematics in first place (34).

The extensive effect of Group II factors may be seen in the fact that one-fourth of the children who make slow progress in school are of normal or superior intelligence (11). Even the community in which a child lives may bear a significant relationship to his success in school. Martens (27) found that comparable groups of urban and rural children were significantly different in school achievement with differences in both reading and arithmetic favoring the former group. Incidentally, in this same study there

was a greater difference in measured school achievement in arithmetic than there was in reading.

That the school is in a good position to ameliorate the damaging effects of some of the out-of-school deterrents to achievement has been demonstrated in a *total push* program described by Coleman (8) in which children were given intensive help for a period of only six weeks. The steps used in the program encompassed not only the creation of a favorable atmosphere for learning but also an attempt to integrate the home environment with the school program. Gains were made in all school subjects with greatest gains in the area of arithmetic. Both Coleman's study and the previously cited study by Martens suggest that the area of arithmetic is especially sensitive to Group II factors.

The general effects of factors such as attitude and out-of-school experience have been seen to have sufficient impact to alter measured intelligence. As McCandless (25) has pointed out, some children develop generalized attitudes unfavorable toward learning almost anything the school has to offer, and other children acquire early habits of thinking in concrete rather than abstract terms. Such children are badly handicapped. Even more severely handicapped are those children who have been afforded "the richness of opportunity to learn self-defeating behaviors" (25: 684).

Group II factors operate to depress and inhibit the realization of full potential at all levels of ability and at all ages.

Group III Factors

These factors arise in and because of the school. They include poor teaching, poor understanding of arithmetic and mathematical concepts and adverse attitudes about arithmetic on the part of teachers, inflexible and unwise grade placement of arithmetical topics, failure to take account of pupils' readiness for learning, and failure to build arithmetical readiness in the early grades.

Several writers (15) (28) have shown that many teachers do not have either sufficient understanding or sufficient skill in arithmetic. Orleans and Wandt (28) tested more than 100 teachers on their understanding of such simple arithmetical processes as

multiplication. One-half of the teachers' answers indicated incomplete or almost total lack of understanding of such processes. For example, almost one-half the teachers missed the following:

"Look at the example below. Why is the third partial product moved over two places and written under the 2 of the multiplier?"

$$\begin{array}{r} 157 \\ 246 \\ \hline 942 \\ 628 \\ 314 \\ \hline 38622 \end{array}$$

The reader will find in Chapter 13 a discussion of inadequate understanding of arithmetic by teachers.

As noted by Brownell (4), when a given topic is arbitrarily placed at a given grade level, many children will have neither the necessary mental equipment nor the prerequisite skills to allow satisfactory progress without special help. In most cases this special help entails reteaching of earlier skills. Teachers are well aware of the problem of the schools' need to consider individual differences, but many lack knowledge of and skill in the necessary techniques and methods for dealing with these differences (15). Lee maintains that our failure in this regard stems from a variety of factors including lack of adequate knowledge of children, insufficient utilization of materials that are available, and failure to use what is known about teaching small groups. Apropos here is Swenson's (35) analogy in which she compares progress in arithmetic by two pupils of very different abilities with distances covered by two cars of different makes and speed. The effect of a single road and a single speed is a reduction in the efficiency of both cars. Similarly, she asserts, two very different children who are forced to travel through the same exercises at the same rate of speed are both unable to realize the achievements of which they are capable.

Finally, the school must assume a major share of the responsibility for failure to capitalize upon many opportunities for building children's readiness for arithmetic. When formal arithmetic instruction begins, especially drill work, many children have not

yet acquired the necessary equipment, attitudes and experiential background. Consequently their initial experiences with number work are unpleasant and they may build avoidance reactions to arithmetic. Perhaps even more unfortunate, pupils may at this point form an adverse attitude toward problem solving in general. The effectiveness of a program to build readiness has been demonstrated in a study by Koenker (21), who compared kindergarten children who had received a readiness program with other kindergarten children who had not. The readiness program consisted of numerous activities such as measuring the room and objects in it, counting, and simple number games. The group which received the early experience with arithmetical concepts were significantly higher on an arithmetic readiness test than the control kindergarten section.

Special efforts to diagnose pupil difficulty and to help pupils learn, as has been so richly demonstrated for reading, may bring about dramatic improvement in arithmetic by alleviating the mistakes previously made in children's early school experiences. Of all the factors that produce differences among children, those which result from poor teaching and programming should be the first to come under attack by arithmetic teachers everywhere.

METHODS OF CARING FOR INDIVIDUAL DIFFERENCES

That individual differences exist is now apparent. Experienced teachers need not be told of the great range and variety of differences. Teachers are constantly being reminded of these differences as they teach, and they are constantly seeking methods for dealing with them. Few teachers, however, are ever satisfied that they are doing a good job in this respect.

At professional meetings and in teachers' discussion groups, the subject of how to care for individual differences is a frequent one, and one of obvious concern. Teachers want and need assistance in finding procedures that will enable them to help a greater percentage of their pupils obtain a satisfactory educational experience.

The remainder of this chapter will be devoted to methods for dealing with individual pupil differences, and to some of the controversial issues related to these methods. Some ways of caring

for individual differences are within the control of the general administration (administrators, school board, and taxpayers) and not within the immediate control of the teacher. These general problems will be considered first. Next, the techniques an individual teacher can use in almost any situation will be discussed.

Administration and School Policy

With proper support from taxpayers, schools can provide help for caring for many of the differences in ability and achievement.

Special Classes. Many schools have separate funds and a special organization for caring for some types of differences. There are classes for the partially sighted, for the deaf, and for many other groups of exceptional children, including mentally retarded but educable children. In a few schools, classes are also organized for the gifted children. In schools with special classes the problems of individual differences will be somewhat alleviated. Unfortunately, even in the most progressive states special classes include only a small percentage of exceptional children. There are not enough teachers or facilities to care for them. Only by aggressively recruiting more special teachers and by providing funds for training and hiring them can this serious problem be solved.

Some schools also have special teachers of arithmetic who use designated activity periods for working with groups of children in need of diagnosis and special instruction programs. Other schools give separate attention to gifted children. For example, the elementary school at Illinois State Normal University has a special arithmetic teacher who works for two 50-minute sessions each week with children in grades 5 and 6 who have been selected as very able in arithmetic. The teacher does not duplicate instruction of the regular classroom, but helps the pupils to be creative with arithmetic and elementary algebra in topics they will not study during their regular classes. One year, for instance, these pupils began an investigation of number notation methods using bases other than ten. This led them to a study of exponents and writing numbers in standard scientific notation. Rather than giving formal instruction, the teacher serves more as an adviser, guiding the pupils and always trying to help them explore and

discover for themselves. These bright pupils gave every evidence of real enjoyment of the work and they grew both intellectually and mathematically as a result of it.

Departmentalization. Another administrative policy that has considerable effect on the problems of individual teachers is that of departmentalization of the elementary school. Where pupils have several teachers during the day, the methods for dealing with individual differences are quite different from those in the self-contained classroom method of organization.

In self-contained classrooms, teachers may use a variety of devices for differentiating the instruction for the various levels of ability and achievement among the pupils. Some of these devices are: grouping within the class, special projects, optional assignments, supplementary problems of a recreational nature, and individual instruction before and after school.

In departmentalized classes, on the other hand, teachers may use many of these same devices, but additional methods are available. For example, homogeneous grouping may be possible in one or two subjects even though the faculty does not desire homogeneous grouping in all subjects. Teachers who have special skills and interest in teaching mathematics as well as the necessary subject background and special knowledge for dealing with individual differences may be assigned the mathematics classes. A teacher of mathematics in a departmentalized school would teach some groups over several years and would thus be able to follow and be responsible for their progress.

Class Size. In some cases, if the teacher is really going to help in the best possible way, the optimum size of the arithmetic class is one pupil. For example, the teacher who diagnoses serious pupil difficulty must observe the pupil's work, ask questions, and penetrate deeply into his background. Such diagnostic procedures require so much time and concentrated attention that the teacher cannot simultaneously give attention to other pupils. In helping the gifted pupil locate appropriate learning materials or work out a problem far beyond the reach of the rest of the class, the teacher must also give concentrated attention to one child. Of course teachers manage to do some of this work during supervised study

periods, at recess, after school, and before school. This kind of help, however, is seriously hampered if there are too many pupils in a class, with the accompanying greater deviation in abilities.

For other experiences in arithmetic, groups as large as 20 pupils are desirable. Discussions, sharing of ideas, group problem solving, competition, inductive method of teaching, using many examples—these all need several pupils in the group to operate successfully.



The optimum size of class for arithmetic instruction is not known, but certainly 25 pupils is near the upper limit, if not beyond it. Taxpayers, school boards, and administrators who expect a teacher to do a good job with all the different children in a class of 35 or 40 pupils are asking the nearly impossible. Most teachers do not have the physical and nervous energy necessary for such large groups, unless they resort to teaching a class as a single group with very little time devoted to individual special problems. One method of dealing with individual differences in arithmetic, therefore, is to keep the size of classes small enough

for adequate teacher attention. This is an administrative problem.

Instructional Materials and Facilities. Proper handling of individual differences among pupils costs money. Cheap education can be provided by lock-step educational methods that attempt to teach large groups of pupils with little attention to the individual. If the educational experience is to help individuals, the instructional program will, of necessity, require more diagnosis, more individual help, more teaching materials suitable for different levels of difficulty, and more teacher time devoted to making plans. Such a program costs money and the taxpayer and administration must assume this responsibility. The teacher can do much to care for individual differences, but support from the community by way of financial assistance must be present.

Homogeneous Grouping. Another administrative decision that has been directed toward solving the problem of individual differences is the decision to place pupils in groups more nearly homogeneous according to some criteria of achievement and ability, or a combination of these and other factors such as motivation, emotional adjustment, and social needs. Of course completely homogeneous grouping is not possible. In this sense the expression *homogeneous grouping* is probably a poor one; it may be more accurate to say *less heterogeneous*. The desire of the administration and the school faculty is not to have a homogeneous group but to have a less heterogeneous group.

Arithmetic is a superstructure of ideas, with one built upon another. As a learner progresses through the subject he gradually goes from a concrete to an abstract level. In the early stages of multiplication, for example, the idea of 3×4 can be established the the use of concrete materials such as three groups of four pencils. The pupil can think that 3×4 means 3 groups of 4 or a total, by adding, of 12. This is concrete and easy to visualize in physical materials. The multiplication problem $\frac{2}{3} \times \frac{3}{4}$ is not easy to visualize or see concretely as $\frac{3}{4}$ added $\frac{2}{3}$ times. In fact, this last statement has no meaning. The pupil now does not refer for his foundation to a material picture of 3 groups of 4 things, but to an abstract generalization: To multiply two fractions,

multiply their numerators and multiply their denominators. Write the products respectively as the numerator and denominator of a fraction which is the product of the original fractions. The pupil is working at an abstract mathematical level, proceeding deductively from the definition of the procedure for multiplying fractions.

As the learner proceeds in his study of arithmetic, the subject becomes more and more abstract. Since new ideas are generalizations and extensions of old ones, the learner must know the concepts and generalizations upon which a new idea is based if he is to understand it. Arithmetic is a cumulative subject in which new learning depends upon old. One cannot learn methods for doing divisions until he understands subtraction; one cannot learn about percent until he understands fractions.

Homogeneous grouping of pupils with differentiation of instruction to fit the level of the group is one method of recognizing the sequential and cumulative characteristic of arithmetic learning. Homogeneous grouping is of itself no help, however, unless the instruction is especially geared to the characteristics of each group that is organized.

Instructional Methods

Except as a voting member of a faculty or of a community, the teacher's primary responsibility for caring for individual differences among pupils is within the classroom. Fortunately, there are many procedures that teachers can use to help meet this responsibility; some of these—grouping within classes, differentiating in pace, differentiating in use of concrete materials, differentiating in use of review and reteaching, and provision of a variety of supplementary teaching aids—will now be discussed.

Grouping Within Classes. Primary grade teachers frequently use grouping within classes in the teaching of reading. The pupils are divided into about three reading level groups and the teacher works with each group independently. Some teachers also establish arithmetic groups, each with a different program. At times the groups may be called together for instruction of the whole class; this is usually done when a new unit is introduced. They



are again separated for differentiated instruction in the latter parts of the unit.

The following description of a unit in adding fractions exemplifies the use of such grouping. The class was called together for initial instruction. As a result of this instruction most of the pupils, except for those few who were too far behind, had a fair understanding that to add fractions one changes them to equivalent fractions with the same number as denominator. The entire class also gave some thought to the process of finding the common denominator.

After working with the entire class on the problem of finding the common denominator when it is present in one of the fractions (for example, $\frac{2}{3} + \frac{1}{6}$, where the common denominator is 6), the teacher divided the class into three groups.

In the slow group the instruction proceeded at a more concrete level, at a slower pace, and with easier problems. The teacher used many visual aids, such as paper plates cut into sectors, rulers, fraction boards, and drawings, to help the pupils make sense in adding such fractions as $\frac{1}{4} + \frac{3}{8}$ and $\frac{3}{8} + \frac{1}{10}$. After instruction of this kind some of the pupils were able to handle problems like $\frac{5}{6} + \frac{3}{4}$, with no emphasis given to finding the least common denominator; the pupils were permitted to find a common

denominator by multiplying 6×4 . No attempt was made to emphasize the fact that while 24 is a common denominator, it is not the least common denominator, although this was discussed if the idea arose naturally.

The fractions used in the slow group were those of the common ruler and of business, with frequent use made of rulers and other materials. Several problems involving real life situations were studied in which the pupils had a chance to apply the ideas of fractions.

For the fast group the instruction proceeded at a more rapid pace. The teacher used concrete materials, but had to refer to them far less often than with the slow group.

The pupils were encouraged to find the least common denominator. They were given addition examples like $\frac{1}{8} + \frac{3}{8} + \frac{3}{8}$, and others that included fractions requiring careful analysis like $\frac{6}{18} + \frac{5}{21}$, to be sure they had obtained the least common denominator. Just for fun the pupils in this fast-learning group were presented the problem $\frac{11}{150} + \frac{13}{315}$, and asked to find the least common denominator. Some found the answer by trial and error. Then the teacher asked if they could invent a way to find the least common denominator. With some hints from the teacher the pupils were led to see that $150 = 5 \times 5 \times 3 \times 2$ and $315 = 7 \times 5 \times 3 \times 3$, so the least common denominator had to be $7 \times 5 \times 5 \times 3 \times 3 \times 2$, or 3150. With one or two pupils in the group the teacher presented the idea of the least common multiple. Although addition of such fractions as $\frac{11}{150}$ and $\frac{13}{315}$ is of little practical value and should not be required teaching, advanced pupils frequently get considerable thrill from being able to handle problems of this type and the teacher has an opportunity to offer a readiness experience for the least common multiple idea, which is used in addition of fractions in algebra.

The middle group reached the stage of learning where they were able to find the least common denominator with some facility even for such problems as $\frac{3}{10} + \frac{2}{5} + \frac{3}{4}$.

As another example of within-class grouping, consider the teacher who introduced the topic of decimal fractions to the class. After the introduction to the meaning of decimals, the class was divided into three groups. In the slow group the teacher had to do

considerable reteaching of material already covered, since memory spans for these pupils are short.

After some checking with the fast group, the teacher posed the question, "Can you write the fraction $\frac{6}{25}$ as a decimal? How would you do it?" The pupils regarded this as a research problem. Of course the teacher could have told them how to do it (and later the class would have instruction on this point), but they were given the problem and encouraged to find the answer any way they could. These pupils felt the challenge that a mathematician feels when he approaches a research problem. They did not know how to work it; they were searching for a procedure; they were being creators rather than simply learners of mathematics.

In describing what the teachers did in the previous examples, we also discussed several techniques which are useful when the pupils are grouped within classes. They are also useful when the class is not grouped. A discussion of these techniques follows.

Differentiation in Pace. In both of the previous examples the teacher differentiated the instruction for the fast and slow groups in several ways, including the pace. In the fast group the pupils moved quickly through elementary aspects of the development and were able to get to a supplementary topic, which might be related to the subject at hand, but at a more abstract level, or it might simply be an interesting but unrelated topic.

In the traditional class organization the teacher can differentiate instruction to a certain extent by permitting a more rapid pace for the faster learners who are anxious to move ahead and are bored if kept working with the same material. However, if the teacher does not want the superior pupils to work ahead in the book, they may, when they finish their immediate lesson, spend the remaining time reading books on arithmetic, working supplementary problems, or trying to discover interesting relationships in mathematics.

Differentiation in Use of Concrete Materials. In the examples given earlier, another means of differentiation involved the use of concrete materials. Teachers have long recognized as important for mathematics the principle of learning *from the concrete to the*

abstract. Successful teaching procedures utilize concrete materials at the early experience level and gradually lead the pupil to abstractions. The end goal of mathematics instruction is the abstract level, because it is at this level that the pupil has power in using mathematics. Fast learners as well as slow learners must start at a concrete level, but the time needed at the concrete level varies. One pupil may be satisfied that there are 27 cubic feet in one cubic yard by looking at a drawing or seeing the teacher wave his finger in the air describing a cubic yard as made up of 27 cubic feet. Another pupil will need to construct 27 little cubes and stack them into a larger cube before he feels secure about this generalization. Differentiation in instruction can thus be accomplished by the length of time concrete materials are used with different pupils. Some will need to cut up several paper picnic plates in the study of fractions; others will need only to think about a few plates with imaginary cuts.

Differentiation in Developing Creativity. In developing discovery or creativity, the teacher presents important related ideas in such a way that the pupil fills in the gaps and actually discovers an important idea for himself. The teacher sets the stage and sees to it that the pupil is ready to discover. With the fast pupil this stage setting can be less completely done than it is for the slow pupil. The slow pupil can discover but he needs more of the important elements or clues brought into his immediate awareness before he can discover.

The teacher can also give challenges to the faster pupils by making leading remarks in class. For example, long before formal instruction in adding fractions with unlike denominators, the teacher might provide the faster pupils with problems that require this operation. Not knowing the formal procedure they will seek ways to solve the problem, and in the process they may discover how to add fractions with unlike denominators. With a little help from the teacher they may even describe the procedure for adding fractions in precise language. The slow pupils will be helped to see this procedure, too, but the teaching will need to be more specific and more concrete.

Differentiation in Review and Reteaching. It was previously pointed out that one of the ways slow pupils differ from fast

pupils is that their memory and attention spans are shorter. Thus it is necessary to spend more time with them in review and reteaching. A program of instruction directed toward helping individuals will be differentiated in the amount and frequency of review and reteaching.

Differentiation in Use of Without-Paper-and-Pencil Arithmetic. The teacher will find that experiences in arithmetic problem solving without the use of paper and pencil is a very good way to interest the more able pupils. For example, the problem might be, "Find the cost of 12 articles if they each cost 19¢." Some pupils will need paper and pencil, but the teacher might ask, "Who can solve this problem without paper and pencil?"

The pupils may invent several ways to do the problem mentally. One may say, "I can solve this by thinking 10×19 is 190 and then 2 more 19's would be 38 more. Altogether that would be 228 cents or \$2.28." Another pupil might say, "If the article cost 20¢, then the total cost of 12 articles would be \$2.40. Since each article is 1¢ cheaper than this, the cost would be 12¢ less than \$2.40 or \$2.28."

When pupils do arithmetic without writing they often free their minds from the formal way of doing the computation and invent ingenious methods. The teacher should encourage this creativity and ask them to do the problems by as many methods as they can. This procedure helps the pupils see important relationships within arithmetic, but more pertinent to the immediate subject of this chapter, it offers interesting and challenging problems for the more able pupils. The teacher can differentiate instruction by having some pupils do the problem one way, some several ways, some using paper and pencil, and some without paper and pencil.

Differentiation through Estimation of Answers. Another procedure that is helpful for all levels of ability is that of having the pupils estimate the answers to their problems by various methods of approximation. The invention of a method of approximating the answer calls for creativity. The able pupil may develop an elegant, ingenious way and should be encouraged to do so, while the slower pupil may depend on more routine methods. The technique of asking the pupils to estimate answers has several

advantages, including the opportunity it provides for differentiating the instruction.

Differentiation Through Solving Problems Several Ways. A similar method of caring for individual differences is that of asking some pupils in the class to try to solve problems in more than one way. This gives the pupil experience in relating one idea to another and thus improves his understanding. Also, some pupils will occasionally develop ingenious methods of solving problems. A teacher might regularly ask some of the pupils to try to solve problems in several ways and give special recognition to the one who develops a particularly good way. By using this technique the teacher has made a contribution to caring for individual differences. The slower pupil has developed at least one way of solving the problems, and has been over the required material. The advanced pupil has had the fun of trying to create several methods and the thrill of inventing ingenious ones.

Other Techniques for Differentiation of Instruction. Grossnickle (17) suggests several other activities that are useful for differentiating instruction and caring for individual differences:

- a. Additional exercises and problems, frequently started because of difficulty
- b. Vocabulary study
- c. Individual and group reports on topics investigated
- d. Finding social applications of mathematics in community resources
- e. Use of supplementary textbooks
- f. Mathematics clubs
- g. Mathematics exhibits and contests
- h. Making models and other visual aids.

Instructional Materials

Teachers of arithmetic can differentiate instruction somewhat in regular classes or grouped classes by use of teaching materials of various kinds.

Arithmetic Corner. Many teachers have found that an arithmetic corner is a big help in dealing with the different levels of achievement. It may contain such items as the following:

1. Interesting books on mathematics such as those listed in

the bibliographies by Hess (18) and Hutcheson, Mantor, and Holmberg (20).

2. Teaching models and manipulative materials for both advanced and slower pupils to use.

3. Computing devices such as an abacus, counting frames, Napier's bones, a small mechanical computing machine, and slide rule, depending upon the grade level of the class.

4. Drill materials such as flashcards, electric drill boards, workbooks, games that emphasize use of arithmetic skills and ideas.

If the teaching materials in the arithmetic corner are to be useful in helping learning at different levels, then the materials must be well selected so that pupils of all levels of achievement in the room will find something stimulating and helpful. In an arithmetic corner for a fifth-grade class, for example, the teacher would probably want sets of flashcards on addition and multiplication for pupils who still need to improve in recall of the basic facts, and computing devices such as an abacus, Napier's bones, simple calculating machine, and possibly a slide rule for those more advanced. Thus pupils at all levels of ability and achievement will find something in the arithmetic corner to hold their interest and provide new learning experiences.

Supplementary Self-Teaching Units

Also useful are supplementary units prepared so that they may be handed to a pupil interested in additional work in mathematics. The unit should be so written that a pupil may read the material and learn it with a minimum of instruction from the teacher. It would have graded exercises and would probably have answers available so the pupil could check his own work. Unit tests would also be provided to help the pupil assess his achievement.

The teachers could work together to prepare some of these units for distribution to classes throughout the school system. The preparation of such units, however, is time-consuming and difficult. Publishing companies could be of service to the schools by publishing such sets of self-teaching units on supplementary or advanced topics in mathematics. Row, Peterson, and Company has already published sets of pamphlets which are a step in this direction, but most of them deal with recreational topics of

arithmetic. It would be possible to have other self-teaching units on advanced topics written in such a way that gifted elementary school pupils could receive much help and inspiration from them.

MAJOR ISSUES IN POLICIES AND PROCEDURES FOR HANDLING INDIVIDUAL DIFFERENCES

Teachers of arithmetic, administrators, and educators are not agreed upon the policies and procedures for caring for individual differences. In fact, on some of the policies and procedures very active arguments take place in faculty, PTA, and school board meetings. Some of the major issues on which the arguments are presented are the following:

1. Some persons are active in supporting a horizontal enrichment of the able pupil's experience in mathematics. These persons are opposed to advancing the pupil to more abstract mathematics or topics that are usually taught in succeeding grades in school. They claim it is possible to provide adequate and desirable learning experiences for the able pupils without advancing vertically. Other persons are strong in supporting the vertical advancement of the able pupils to more advanced mathematics. They believe the failure to advance causes boredom and poor work habits on the part of the able pupils (30).

2. Some persons are opposed to the homogeneous grouping of pupils within a grade level. They claim that homogeneous grouping really is impossible, that it is non-democratic, and that it is hard on a pupil's mental health. Other persons are in favor of grouping pupils on the basis of achievement and ability; they argue that mathematics is a sequential subject that requires understanding of basic ideas at one level before learning can proceed to a next level. They deny that homogeneous grouping is non-democratic or a cause for poor mental health.

3. Another basic issue concerning the problem of individual differences has to do with promotion policy. Some persons urge that promotion in school be based almost solely on achievement results. They blame "social promotion" as a major cause for the wide dispersion of talent in some of the upper grades of the elementary schools. Others are in favor of a promotion policy

that includes achievement as one of the factors, but also includes social age-level adjustment as one of the major factors.

4. Another issue concerns strict adherence to the grade placement of topics. Some persons are in favor of setting standard grade level objectives in mathematics and then requiring that all pupils meet these standards. The teacher should not go beyond these standards, they state, because of the administrative difficulties that result. Other teachers and administrators favor rather flexible grade level standards, set according to the ability of the class. In some cases this means that the class will proceed to a study of more advanced topics, and in other cases it will mean that the class will not complete the usual work of some standard curricula in mathematics.

5. There is a polarity of views toward the type of study that should be provided the able pupil in mathematics. Some persons favor supplementary work for these pupils whereby they would gain experience in the social and economic applications of mathematics. They maintain that school experience contains too little reference to the relationships among the major disciplines and the ramifications in general education. Others believe that social-economic applications are both inappropriate and boring for the bright pupil, who needs work in abstract mathematics.

Very little definite research exists on the above arguments, and teachers will not find it possible to choose a position solely on the basis of research findings. Some of these issues probably can never be settled on a purely empirical basis, either, for basically the arguments are philosophical in nature. Experiments and surveys would be of considerable help in providing facts, but the final decision rests upon assumptions made in one's educational philosophy.

Some school faculties and administrations have well-defined policies, and the individual teacher can work as a member of the faculty to modify or support these policies. However, the principal approach of the teacher will be to adjust to existing school policy and work out the best program of instruction within this policy.

Whatever the over-all school policies, it is still the individual

classroom teacher who must adapt instruction to the differences he finds among his pupils. The teacher can be encouraged by the knowledge that there is a growing variety of instructional materials to assist him in his task. Finally, it should be heartening to note the growing awareness of this problem among both professional and lay groups.

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Guidance and Counseling

IRENE SAUBLE and C. LOUIS THIELE

UNTIL COMPARATIVELY recent years guidance and counseling activities were looked upon as functions for high schools and colleges. Today increased attention is being focused upon the development of guidance programs which extend from the kindergarten through all the years of a pupil's life in school. It is recognized that adequate attention to the needs of pupils in their early, formative years will serve to prevent serious scholastic and behavior difficulties later in their school careers. The need for guidance does not suddenly become evident when the elementary school is left behind. Clearly-defined guidance programs at the lower levels, like adequate health programs, will result in more positive benefits at the secondary school level.

To the extent that space permits, this chapter will deal with the following aspects of guidance and counseling in the elementary school: (a) generally accepted basic concepts, (b) requirements for an effective program, (c) the role of a testing program, and (d) implications of such a program for guidance in the field of arithmetic.

GUIDANCE CONCEPTS FOR ELEMENTARY SCHOOLS

The broad point of view regarding guidance is very well stated by Traxler (2:2):

Ideally conceived, guidance enables each individual to understand his abilities, interests, and personality traits, to develop them as well as possible, to relate them to his life goals, and finally to reach a state of complete and mature self-guidance as a desirable citizen of a democratic social order. Guidance is thus vitally related to every aspect of the school: the curriculum, the methods of instruction, the supervision of instruction, disciplinary procedures, attendance, problems of scheduling, the extracurriculum, the health and physical fitness program, and home and community relations.

That the problems involved in establishing effective guidance and counseling programs are highly complex is evident from Traxler's statement. We may well identify certain concepts for consideration in relation to the elementary school program.

All children need guidance if they are to discover and develop their potentialities to the maximum extent. This concept is forcefully stated in the 1955 Yearbook of the Association for Supervision and Curriculum Development (1:12):

If all boys and girls were equally alert and vigorous, equally intelligent, adjustable, and interested in school learning, there would be no case for curriculum flexibility and far less need for guidance. The knowledge that each child presents a unique pattern of characteristics and requires unique treatment lies at the heart of our guidance philosophy. With individual differences as our starting point, we view guidance as relating to all those things which adults do *consciously* to assist an individual child to live as fully and effectively as he is able.

To be effective, guidance must be inherent in the teaching process. Instruction must be inseparable from guidance. The teacher must guide as he teaches. To make this principle operate, the instructional program must be sufficiently flexible to enable the teacher to make adjustments in both content and methods to meet the needs of pupils whose ability levels range from the slowest learners at one extreme to the most rapid learners at the other extreme.

Teachers who integrate guidance with instruction develop skill

in obtaining, organizing, and utilizing significant and accurate information about each individual pupil.

The team approach is essential to a functional guidance program. Although the classroom teacher in elementary school may be regarded as the key person in the guidance process, this does not minimize the importance of the contributions of other school personnel and of parents.

Guidance includes personal and social areas as well as vocational and educational planning.

REQUIREMENTS FOR AN EFFECTIVE GUIDANCE PROGRAM IN THE ELEMENTARY SCHOOL

Although the patterns for guidance programs at the elementary level vary widely, the literature in this field and in the related fields of mental hygiene, child development, and curriculum contains many specific suggestions concerning the essential features of an ideal program. With the general guidance concepts as background, the requirements for a recommended program will be outlined and discussed briefly.

A Guidance-Oriented School Principal

The success or failure of a guidance program is dependent upon the school principal since he provides leadership in the development of the philosophy and policies of his school. He is responsible for initiating and carrying on an inservice training program to develop teachers with the guidance viewpoint. Even when the services of a school counselor are available, the principal must insure that an organized program of guidance evolves.

Specially Trained Guidance Workers

There is a definite trend toward the employment of guidance specialists in elementary schools. They may serve an individual school or a group of schools. Their training enables them to work intensively with complex cases referred to them and they can often bring about modification of unfavorable school conditions, or help resolve emotional conflicts.

Some of the specific services to be performed by a trained elementary school counselor are the following:

1. Take charge of the guidance testing program within the school.
2. Assist in organizing and maintaining pupils' cumulative records.
3. Assist teachers in the effective use of all types of test results.
4. Work with parents in obtaining information needed in the adjustment of the child and in interpreting to parents the measures taken by the school for the best interests of the child.
5. Recommend curriculum adjustments for individual children and assist in putting these adjustments into operation.
6. Assist in the utilization of community resources which contribute to more effective adjustment of pupils.

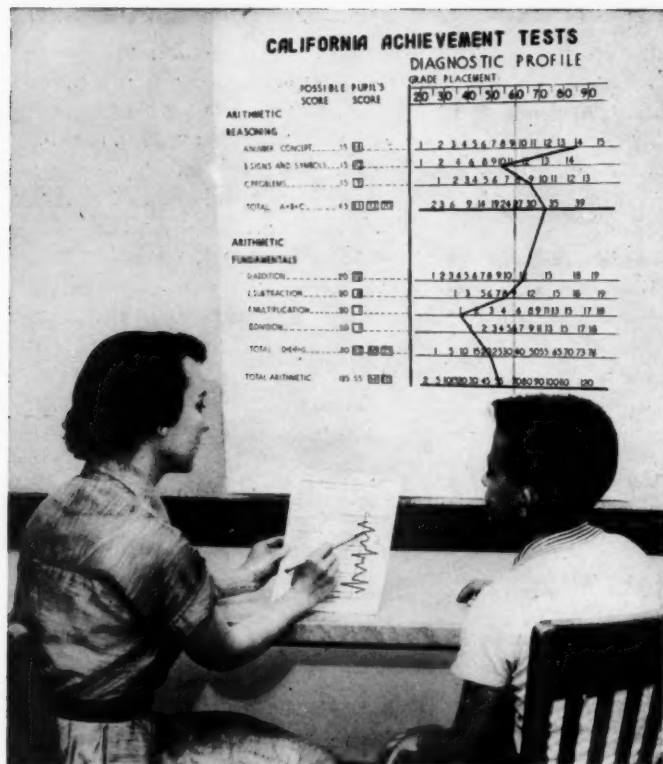
Guidance-Minded Teachers

Ideally, all teachers should have some training in guidance. More important, however, than specific training in guidance techniques and procedures is the teacher's point of view with regard to subject matter aims and the development of children. The teacher who places mastery of subject matter above all else is not likely to be sensitive to differences in individual capacities and interests of his pupils. Study of his pupils is often restricted to a determination of how much or how little they know in different subjects, and he is not likely to sense their feelings or discover the reasons for their reactions to learning situations.

An elementary school possesses some of the essential requirements for a successful guidance program if its teachers have developed effective teaching skills, if they are thoughtful and considerate, and if they understand their pupils and plan accordingly.

Assistance of Other Specialists

A successful guidance program requires the services of a staff of specialists—the doctor, nurse, school social worker, psychologist, and psychiatrist—who serve the entire school system. These specialists are needed to supplement the work of school personnel and of parents when pupils present special problems of health,



physical disability, or severe behavior and emotional maladjustment.

The Cumulative Record System

A comprehensive, well-organized cumulative record system is indispensable for the effective guidance of children. The general characteristics of such a system are the following:

1. A cumulative record should be started for each child at the time of his entrance in school. It should be transferred with the child as he progresses from grade to grade, from lower to higher

schools, or moves to another school district. It should be so maintained that information is accurate, complete, and up-to-date.

2. The record should reveal a pupil's growth over a period of years in relation to his own capacity, in relation to other pupils in his class, and in relation to children of his own age group.

3. Cumulative records should be flexible enough to provide certain uniform data for all pupils, but should allow for as much additional data as seems feasible for individual pupils.

4. Cumulative records should be kept in the school office and should be readily accessible to the entire faculty. The only value in keeping records is to have them used. In instances where information about a pupil is too confidential to be available to all teachers, this information should be filed in a separate place.

5. A folder or envelope is probably the most advantageous form for the cumulative record. Data may be recorded on both sides of the folder or envelope in specially prepared blanks. Many types of information about pupils may be placed in the folder to supplement that recorded on the outside.

6. While the material placed in the cumulative record may vary from school to school, the following summary includes the information needed by guidance workers:

Personal information—enrollment and attendance, home and community information; health data

Scholarship—school marks by years and subjects; special reports on failures; record of reading, of study habits, of citizenship growth

Test scores and ratings—general intelligence-test scores; achievement-test scores and other test scores; personality ratings

Special interest data—hobbies, educational plans, vocational interests; co-curricular activities, work experiences; other special interests

Supplementary material—anecdotal records, case studies; teacher's or counselor's comments; follow-up data.

Instructional Materials

If teachers are to guide as they teach, they must have available an adequate supply of instructional materials of superior quality,

[illegible]

they should be administered in a recommended testing program for kindergarten through grade eight.

It will be noted that group intelligence tests are scheduled at four-year intervals. Standardized achievement tests provide for the measurement of growth in basic skills over two-year periods. Because of the relatively large amount of testing time required for the entire battery of achievement tests, one-year intervals are not recommended. Also, the amount of growth in each area tested can often be determined more reliably over a two-year period.

Interest inventories given in the fourth, sixth and eighth grades assist guidance specialists and teachers in determining changes in pupils' interests and aims as they gain experience and maturity. They also provide a basis for comparison of the long-range educational goals of the pupil with his ability and achievement status.

For excellent annotated lists of tests of all types, the reader is referred to *Techniques of Guidance*, revised edition, by Traxler, pages 59-72 and 84-87.

The achievement test results should be summarized graphically on individual pupil charts. Such a chart is illustrated in Table 2, which shows a hypothetical profile formed by plotting the grade equivalents for the Iowa Tests of Basic Skills, Multi-Level Edition which had been given in grades four, six, and eight.

General Uses of Data from Achievement Tests

The information obtained from the periodic administration of achievement test batteries should have the following uses for guidance purposes:

1. Provides an objective record to indicate how well certain skills have been mastered regardless of grade placement, report card marks, or learning aptitude.
2. Measures the growth in skills tested over regular periods of time.
3. Indicates to the teacher and the counselor the type and amount of adjustment that must be made in teaching a class with a wide range of grade-equivalent scores.
4. Assists school personnel in helping parents to gain an objective evaluation of their child's achievement.

TABLE 2

PUPIL PROFILE CHART

F ☐ M ☐

Birth Date

S A L L Y

First

J O H N S

Last Name

The Three Lines Below are for Grade Equivalent Scores on the Iowa Tests of Basic Skills, Multi-level Edition

	V	R	L1	L2	L3	L4	L	W1	W2	W3	W	A1	A2	A	G
86	86	86	90	83	95	96	92	85	83	90	86	88	84	86	87
77	77	68	70	60	68	72	67	67	67	74	69	68	61	64	69
54	54	49	54	48	48	51	50	45	43	41	43	47	48	48	49

RECORD OF TESTING

8th GRADE

Date: Nov. 1958

School: _____

Form Used: 1

8th GRADE

Date: Nov. 1956

School: _____

Form Used: 2

4th GRADE

Date: Nov. 1954

School: _____

Form Used: 1

KEY

V - Vocabulary

R - Reading

L1 - Spelling

L2 - Capitalization

L3 - Punctuation

L4 - Usage

L - Total Language

W1 - Maps

W2 - Graphs

W3 - References

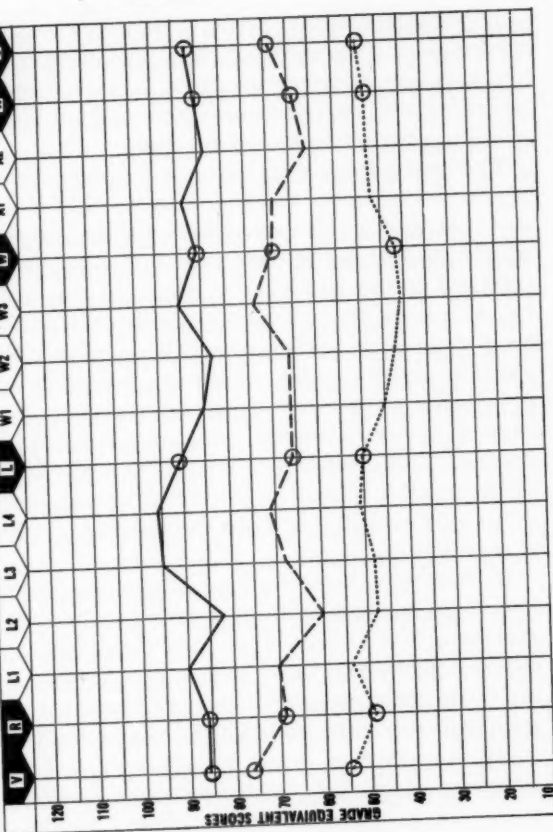
W - Total Work Skills

A1 - Concepts

A2 - Problems

A - Total Attitudes

C - Composite - All Scores



GUIDANCE AND COUNSELING IN ARITHMETIC

In preceding sections we have discussed the status of guidance programs at the elementary level and identified basic concepts. We have set up requirements for a recommended program and described in some detail one of the essential features—a comprehensive system-wide testing program.

The remainder of this chapter will focus attention upon the use of effective guidance procedures in the field of arithmetic, to the end that each pupil may receive training in mathematics commensurate with his interest and ability. The urgent needs of the technical and scientific age in which we live demand that we attain this goal. We shall presuppose that the services of a specially trained guidance worker are available, or lacking a person designated as counselor, that the school principal or assistant principal is qualified to perform the duties of a counselor. We shall presuppose guidance-minded classroom teachers who are skillful, ingenious, and courageous enough to make adjustments in the subject-matter content of arithmetic and in teaching techniques in accordance with the individual abilities, interests, and needs of their pupils.

In the practical aspects of curriculum planning and in consideration of the learning abilities of pupils, we usually plan a body of experiences for average pupils, and then make differentiations for superior pupils and for slow learners. Accordingly, instructional materials, teaching procedures, and standards of achievement in arithmetic are better adapted to the learning levels and interests of the average pupil in our schools than to the superior pupil or the slow learner. For this reason the discussion of guidance and counseling in arithmetic will consider first the special needs of pupils whom we classify in these groups.

Guidance in Arithmetic for Superior Pupils

Identification of Superior Pupils. It is of paramount importance that scholastically gifted pupils, who constitute between 2 and 3 percent of the entire school population, be identified as early as possible in their school careers. For the identification of this group it is necessary to use all the data in the pupils' cumulative

record as well as information from other sources. According to recent literature on the subject, a pupil may be considered gifted (a) if he has an intelligence quotient of 130 or above, (b) if his scores are two years or more above grade level in all areas tested by standardized achievement tests, (c) if his school marks have been consistently above average, and (d) if he has been judged by school personnel to possess attributes which are characteristic of superior pupils.

It has been pointed out that group intelligence tests may not always be accurate, and that individual intelligence tests administered by a trained psychologist may be necessary to locate some gifted pupils. Likewise, achievement test scores and school marks may not be entirely reliable criteria. However, there seem to be certain attributes which are generally accepted as evidence that a pupil belongs among the gifted. Among the characteristics most frequently listed are the following:

1. Extraordinary memory
2. Intellectual curiosity
3. Ability to do abstract thinking on a high level
4. Ability to apply knowledge to other situations
5. Persistence in worthwhile behavior
6. An outstanding degree of originality, resourcefulness, initiative, and imagination
7. Organizing ability and leadership qualities
8. High capacity for self-appraisal and self-direction.

It must be remembered that pupils whom we designate as gifted differ from the group often called superior largely in the degree to which they possess the characteristics described. Many pupils with IQs of 120 would fall into this classification of superior. In establishing an effective guidance program in arithmetic, it is essential to plan learning experiences appropriate for both groups. In this discussion we shall use the term *superior* rather than *gifted* to designate the pupils under consideration.

The Services of the Counselor. The counselor's initial responsibility in cooperation with the classroom teacher is the identification of superior pupils. A systematic method must be devised for keeping track of them and for making certain that their special

needs are met. In some schools an appropriate sticker is placed on the cumulative record folder.

The counselor should arrange conferences with the parents of mathematically superior pupils. It is important to know whether the parents are setting educational goals which are commensurate with the pupils' high potential. In some economic areas, parents are unaware of their child's high ability level and their aspirations do not envisage a college education. These parents need the assistance of the school counselor who can provide information about the means of obtaining college training for superior children even when parents are financially unable to provide it.

Conferences with parents who recognize their child's special mathematical ability and who have high aspirations for the child are also important. Through the cooperation of the school and the home, a wide variety of community resources may be utilized for the enrichment of the child's out-of-school experiences. If acceleration seems desirable, the counselor may suggest attendance at a summer session.

The counselor must give consideration to another type of capable pupil—the one whose achievement in arithmetic falls far below expectancy on the basis of his learning aptitude. It is sometimes pointed out that pupils with high potential for arithmetic lose interest and become poor achievers because the learning experiences have been unsuited to their capacity. Uninspired teaching, excessive amounts of routine drill, and a dearth of functional applications contribute to loss of interest in arithmetic. If these are the factors causing low achievement, the remedy lies in assisting the teacher to the point that differentiated experiences such as those described in the next section are provided for the pupil.

Low achievement in arithmetic may be due to a combination of circumstances which involve the home as well as the school. Some parents provide a high degree of stimulation for pupils in the development of communicative skills and in all language arts, but neglect to provide equally high motivation for pupils in the development of quantitative concepts and skills. Able pupils must recognize the need for the development of a high degree of competence in mathematics.

A different type of problem confronts the counselor and the

teacher when a capable pupil is a non-achiever in arithmetic because he is emotionally and socially maladjusted. This problem is extremely complex and cannot be covered in this chapter. Only problems of educational adjustment can be included.

Organizational Practices for Meeting the Needs of Superior Pupils. The principal, counselor, and teachers must cooperate in deciding upon the most effective organizational practices within the school to meet the needs of these pupils. Practices which may be considered include: (a) ability grouping, (b) acceleration by double promotion, and (c) an enrichment program for superior pupils within the regular classroom.

Some large schools find it feasible to group pupils at each grade level into two groups—those capable of moving ahead rapidly in the academic subjects, and those for whom a slower pace is desirable. In smaller schools a similar type of grouping may be made on the basis of two grade levels. This type of ability grouping has the following advantages:

It provides a greater stimulation for the more capable pupils, since they work with an entire group of keen minds.

It prevents capable pupils from becoming bored and lazy, as they sometimes do in the regular classroom.

It provides increased opportunities for enrichment, since the teacher can adjust the work to a narrower range of ability and achievement.

Some acceleration as well as enrichment can be achieved, since the entire group of capable pupils will profit from the additional motivation and the larger share of the teacher's time which they receive. However, acceleration should not consist of mere ground-covering of a narrowly-conceived body of arithmetic content. It should not be accompanied by a feeling of undue haste to move from topic to topic. Instead, each arithmetic topic should be explored to the depth and extent that seems appropriate for the developmental level of the pupils.

Double promotion is a practice sometimes employed to achieve acceleration of the superior pupil. If no serious gaps are to be left in the pupil's knowledge and skills, some systematic plan must be made for the pupil to master the work of the grade skipped. In school systems which operate summer sessions, provision may

be made for the superior student to do the work of an entire semester during the summer.

Since neither ability grouping nor double promotion can be practiced generally, we shall give greatest consideration to a differentiated program in the regular classroom as a means of meeting the needs of the superior pupil in arithmetic.

Curriculum Adjustments in Arithmetic for the Superior Pupil. Adjustments in the arithmetic program must be made to meet the needs of superior pupils whether they are taught in a specially organized class or as a small group within the regular class. The guidance-minded teacher takes into consideration the characteristics of these pupils as he makes the necessary adjustments in content and in teaching procedures.

Superior pupils possess the ability to form generalizations and to move quickly to abstract levels of thinking. Accordingly, these pupils require a minimum amount of work on the concrete level during the developmental phase of new learning. The teacher observes their reactions, questions them individually or as a group, and makes certain that they do not waste time and effort working on an immature level when they are ready to move to more mature levels of thinking.

Let us illustrate with subtraction of fractions and mixed numbers when borrowing is first introduced in grade 5, as in examples (a) to (f).

(a)	(b)	(c)	(d)	(e)	(f)
1	2	4	$3\frac{1}{2}$	$5\frac{1}{2}$	$5\frac{1}{2}$
$-\frac{3}{4}$	$-\frac{3}{4}$	$-1\frac{1}{4}$	$-\frac{1}{2}$	$-3\frac{1}{2}$	$-2\frac{1}{2}$
<hr style="width: 100%;"/>	<hr style="width: 100%;"/>	<hr style="width: 100%;"/>	<hr style="width: 100%;"/>	<hr style="width: 100%;"/>	<hr style="width: 100%;"/>

When the teacher plans to have the entire class participate in the developmental lesson, he will have available a supply of representative materials which pupils may utilize in finding solutions if they need help. For this lesson, there may be paper plates marked off into fractional parts and parts cut from flannel for use on the flannel board.

The teacher will observe that a superior pupil uses manipulative materials for the solution of one or two examples. He may work examples (b) and (d) by using materials. For another example he may work with diagrams of circles, dividing one circle into

parts of the size that are to be subtracted. From that point on, he imagines the change in one whole and performs the subtraction with symbols. He has grasped the basic idea; he has generalized. This pupil would be handicapped and bored if he were required to continue to obtain solutions by cutting up wholes and removing parts.

Some pupils may be able to perform these subtractions correctly without reference to manipulative or diagrammatic aids. They may be able to transfer ideas learned in the process of borrowing with whole numbers to this borrowing with fractions. With whole numbers, however, the borrowed one is always transformed to ten of the next smaller unit: one ten to ten ones, one hundred to ten tens, etc. In subtraction of fractions, the borrowed one is transformed into parts of the size that are to be subtracted. Accordingly, it is the denominator of the fraction which indicates the number of parts into which the whole is to be divided. The teacher's responsibility to the superior pupil is to provide the algorism for recording his thinking and to help him in expressing the generalization he has discovered in correct, precise language.

Often the superior pupil finds it unnecessary to place any work for an example on his paper. When he has gained understanding he thinks and writes only the answer, or he omits certain steps which he considers superfluous. Individual pupil-teacher planning is necessary to decide when the steps in the solution should be shown and when they may be omitted. Capable pupils should hold high standards for themselves in habits of work, accuracy, and in the development of mature ways of thinking.

The superior pupil requires practice material in much smaller amounts than that needed by less capable pupils. He is irritated by heavy assignments of routine drill. The teacher must make differentiated assignments both in the type and amount of drill. Soon after a new generalization and new computational procedures are understood, the superior pupil is ready to engage in cumulative practice in which he mixes both new and previously-acquired learnings. Even these pupils need adequately motivated and properly spaced review exercises to help in the maintenance of the high levels of performance for which they have the potential.

The superior pupil is capable of developing a high degree of

resourcefulness. He should be encouraged to use a variety of ways of finding answers and to use one way as a check upon the correctness of a solution obtained by another method. Some illustrations are given.

In the learning sequence for subtraction of fractions and mixed numbers, examples which involve both borrowing and changing to a common denominator are presented. The usual algorism and the solution for such an example is shown below.

$$\begin{array}{r} 3\frac{2}{3} = 3\frac{4}{6} = 2\frac{10}{6} \\ -1\frac{5}{6} = -1\frac{5}{6} = -1\frac{5}{6} \\ \hline 1\frac{5}{6} = 1\frac{1}{2} \end{array}$$

To illustrate the resourcefulness of one superior pupil, consider his method of solution for this example. He actually did not need to record any of the steps of his thought process. If he had recorded them, they would have appeared as below.

$$\begin{array}{r} 3 \qquad 1\frac{1}{6} \\ -1\frac{5}{6} \qquad +\frac{2}{3} \\ \hline 1\frac{1}{6} \qquad 1\frac{5}{6} = 1\frac{1}{2} \end{array}$$

This pupil considered the 3 in the minuend and subtracted $1\frac{5}{6}$ from 3, obtaining $1\frac{1}{6}$. Next he added the $\frac{2}{3}$ of the minuend, which he had disregarded, saying $1\frac{1}{6} + \frac{2}{3} = 1\frac{5}{6}$ or $1\frac{1}{2}$.

When challenged by other pupils in the class to explain his method of solving the example, he did so by getting $3\frac{2}{3}$ paper plates and he visualized each step for them in the sequence described above.

In the following exercise pupils are asked to experiment with methods of checking answers for multiplication and division examples. The questions are given in the column at the left and the solutions are shown at the right.

In example a, would the product be changed if you multiplied first by the 3 in hundred's place, next by the 2 in ten's place, and then by the 9 in one's place? Why or why not? Prove your answer.

	a	456
		$\times 329$
456		1368
329		912
		4104
		<hr/> 150024

In example **b**, can you check the product by multiplying 548 by 100 and then subtracting 2 times 548? Why? Prove it.

$$\begin{array}{r} 548 \\ \times 98 \\ \hline 4384 \\ 4932 \\ \hline 53704 \end{array}$$

b

$$\begin{array}{r} 54800 \\ -1096 \\ \hline 53704 \end{array}$$

In example **c**, can you check the product by first multiplying 286 by 7 and then multiplying that product by 7? Why?

$$\begin{array}{r} 286 \\ \times 49 \\ \hline 2574 \\ 1144 \\ \hline 14014 \end{array}$$

c

$$\begin{array}{r} 286 \\ \times 7 \\ \hline 2002 \end{array} \quad \begin{array}{r} 2002 \\ \times 7 \\ \hline 14014 \end{array}$$

In example **d**, can you check the quotient by first dividing 2664 by 6 and then dividing the quotient you get by 6? Why? Prove it.

$$\begin{array}{r} 74 \\ 36 \overline{)2664} \\ \underline{252} \\ 144 \\ \underline{144} \\ 0 \end{array}$$

d

$$\begin{array}{r} 444 \\ 6 \overline{)2664} \\ \underline{6} \\ 6 \\ \underline{6} \\ 0 \end{array} \quad \begin{array}{r} 74 \\ 6 \overline{)2664} \\ \underline{6} \\ 6 \\ \underline{6} \\ 0 \end{array}$$

Superior pupils should have many exercises in which the emphasis is placed upon reasoning, analyzing, and applying generalizations. An illustration is given.

In the following oral exercise, examples with answers are given. The pupil is asked to decide which answers are incorrect, without using paper and pencil to work the examples, and to explain in detail the reasoning which was used.

Examples

$$24 \div .75 = 3.2$$

Reasoning

INCORRECT. The quotient would have to be larger than 24 because the divisor (.75) is smaller than 1.

$$2.9 \times 38.6 = 103.94$$

ANSWER IS REASONABLE. 2.9 is about 3; 38.6 is about 40; $3 \times 40 = 120$.

$$.26 \times 85 = 221$$

INCORRECT. .26 is about $\frac{1}{4}$; $\frac{1}{4}$ of 80 = 20, so the correct answer is about 20.

$$\frac{1}{3} \times \$18.40 = \$20.70$$

INCORRECT. The product would have to be less than \$18.40 because the multiplier ($\frac{1}{3}$) is smaller than 1.

$$45 \times 8 \times 0 = 360$$

INCORRECT. The product is zero because one factor is zero.

$$600 \times 875 = 52,000$$

INCORRECT. $600 \times 800 = 480,000$. This is a 6-place number, so a 5-place number could not be correct.

$$5\frac{1}{4} \times 80 = 460$$

CORRECT. $6 \times 80 = 480$
 $\frac{1}{4}$ of 80 = 20
 $480 - 20 = 460$

$$16 \div \frac{1}{4} = 21\frac{1}{3}$$

CORRECT. $16 \div \frac{1}{4} = 64$
 $64 \div 3 = 21\frac{1}{3}$

$$15,792 \div 56 = 2,820$$

INCORRECT. $15,000 \div 50 = 300$. The quotient could not be a 4-place number.

Superior pupils should be challenged daily to estimate answers and to obtain exact answers without paper and pencil. In some of the above examples, the pupil rounded the numbers and obtained an estimated answer as a check upon the reasonableness of the given answer. When pupils develop ability in recognizing number relationships and in using short cuts, they achieve remarkable proficiency in finding exact answers by mental computation.

Word problems for superior pupils should be increased in complexity and may be based upon a greater variety of social situations than for less capable pupils. Data may be collected and organized by the pupils, and they may formulate their own problems with teacher guidance. As an illustration we may consider the vast amount of problem data which pupils may obtain from a study of travel schedules for buses, trains, planes, and boats. Problems may be formulated and solved which deal with the length of time required for trips, comparative costs by different modes of travel, rates of speed, time zones and other aspects of travel.

Social studies materials are a rich source of problem data. The effective use of scales and symbols on maps and globes requires mathematical understanding and skill. General language units may stimulate the study of foreign currencies.

Excellent enrichment in arithmetic for all pupils will result if the teacher guides the superior pupils to conduct research on selected topics and make reports to the entire class. Appropriate topics include the following:

- Development of the Hindu-Arabic number system
- History of the Roman number system with emphasis upon its disadvantages for carrying out computations
- Historical development of clocks and watches
- Development of standard time and its importance to people in radio, television, and transportation
- Development of the 24-hour clock used by military and naval authorities
- The stock exchange and its operations
- Scales of notation with bases other than ten
- Unusual ways of multiplying—lattice method, doubling method and lightning method
- Historical development of measurement
- Egyptian and Roman systems of fractions
- Historical development of decimal fractions
- Longitude, latitude and time zones.

Recreational arithmetic of various kinds appeals to the superior pupil: magic squares, number puzzles, number tricks, and short cuts. After practice in working some puzzles, pupils can often construct original ones to exchange with other pupils. Many teachers assemble a file of recreational arithmetic materials which they make available for pupils to use after they complete regular assignments.

Summary of Goals in Arithmetic for Superior Pupils. Curriculum adjustments in arithmetic should enable superior pupils to attain these goals: (a) deeper insight into the logic of the number system, (b) knowledge of the historical development of numbers and measures, (c) increased facility in mental computation and use of shortened computational procedures, (d) greater appreciation

of the importance of mathematics in our culture, (e) real enjoyment and satisfaction in the study of mathematics.

Guidance in Arithmetic for the Slow Learner

Identification of the Slow Learner. In some respects the term *slow learner* is a relative one. In a class with a predominance of superior or high-average pupils, the teacher is likely to consider pupils of average ability as slow learners. In the literature of guidance, however, the designation *slow learner* is reserved for pupils with IQs between 75 and 90. They are regarded as having ability levels high enough to justify keeping them in the regular classroom, but they usually have pronounced difficulty in keeping up with the average speed of the class in academic work.

Although some slow learners will be found among children known as discipline cases, they need not develop into problem children if the proper preventive measures are taken by the school and the home. It is most unfortunate that some teachers expect slow learners to achieve at the level of the average pupil and rate them as failures when they do not measure up to these standards. Such a teacher may not realize that the pupil is making a desperate effort to succeed and is doing as well as should be expected in view of his below-average capacity. In this type of situation after repeated failures the slow learner may become discouraged and hurt and may stop trying. He may become sullen, hostile, and aggressive. He may gain attention by fighting, arguing, and causing all the trouble he can for the teacher.

The Counselor's Role in the Guidance of the Slow Learner. It is the responsibility of the counselor to assist the teacher in identifying the slow learners in his class. From information in the cumulative record—test results, anecdotal records, results of interviews, and psychological examinations—he must discover as much as possible about their personal and social needs as well as their educational needs.

The counselor must also work with the teacher to help him understand the limitations of the slow learner. The teacher must accept the slow learner as he is and realize that he should not be blamed for his mental insufficiency. Slow learners need to ex-

perience success on their level just as much as average and above-average children do.

A satisfactory program must involve all the teachers in a school and not just one or two who may be particularly sympathetic to the needs of slow learners. The principal and counselor must coordinate the efforts of teachers who make the adjustments discussed in the next section. If a teacher of grade four helps a pupil to cover, in a manner satisfactory for him, only half the work of the grade, it is evident that the next teacher must be willing to adjust the work of his grade accordingly. Textbooks, workbooks, and learning aids designed for lower grades must be made available to teachers who try to plan educational experiences profitable to slow learners.

If the school organization makes ability grouping possible, the counselor sees that slow learners are placed in the classes where they can work most advantageously. The grading and reporting system must be worked out carefully with teachers. To provide slow learners with some degree of satisfaction, they must be judged in terms of their effort and in comparison with their own previous records rather than with the average for the class.

The counselor works with the parents and helps them to see that the child's low achievement is not because he is lazy, mean, or completely disinterested. Sometimes a slow-learning child is rejected and misunderstood by his family. This is particularly true when there are average or superior children in the family and the slow learner is expected to maintain the achievement level set by the other children. The family as well as the school must learn to place more emphasis upon what the slow learner can do rather than upon what he cannot do.

There are some school services which slow learners are capable of performing. The counselor will make a practice of selecting slow learners to distribute school supplies, to count and arrange books, to weigh and measure quantities of materials collected in paper, magazine, and clothing drives, and so on. These activities will help to give them recognition and status.

Curriculum Adjustments in Arithmetic for the Slow Learner.
The friendly, conscientious teacher who accepts the slow learner

makes practical plans to prevent discouragement in arithmetic. He takes into consideration these specific needs of the slow learner:

- Increased opportunities to work with representative materials
- Provision for large amounts of drill to fix new learning
- Closely-spaced cumulative review exercises to combat forgetting
- Greatly reduced pace of learning necessitated by the fact that learning must progress by short steps
- Variation in activities because of short attention span
- Continuation of work on immature levels for extended periods of time because of slow learners' inability to recognize relationships or to generalize readily
- Application of arithmetical processes to relatively simple word problems because of pupils' restricted reading ability and inability to comprehend complex situations.

Adjustments in arithmetic in harmony with the above needs will be illustrated for different grade levels.

GRADE 3. SUBTRACTION OF TWO- AND THREE-PLACE NUMBERS. The introductory steps of this unit include the solution of examples such as those below.

$$\begin{array}{r} 20 \\ -8 \\ \hline \end{array} \quad \begin{array}{r} 51 \\ -9 \\ \hline \end{array} \quad \begin{array}{r} 60 \\ -18 \\ \hline \end{array} \quad \begin{array}{r} 84 \\ -16 \\ \hline \end{array}$$

The slow learner will need many experiences in working with manipulative materials and in drawing pictures of imagined actions to gain the concept of place value, and to learn how to regroup one ten as ten ones to make subtraction possible.

The teacher may need to guide the slow learner in the use of a variety of representative materials, such as the following: (a) real coins—dimes and pennies—changing one dime to pennies, (b) 10-bundles of straws and single straws, (c) a pocket chart with tens' and ones' places indicated.

Writing out the change in words often proves helpful:

$$\begin{array}{r} 84 = 8 \text{ tens } 4 \text{ ones} = 7 \text{ tens } 14 \text{ ones} \\ -16 = 1 \text{ ten } 6 \text{ ones} = 1 \text{ ten } 6 \text{ ones} \\ \hline 6 \text{ tens } 8 \text{ ones or } 68 \end{array}$$

$$\begin{array}{r} 70 \\ 84 \\ -16 \\ \hline 68 \end{array}$$

The teacher encourages the slow learner to show the changes in the minuend until such a time as he feels secure and can operate without these so-called crutches.

Ample amounts of practice on each new learning must be provided. Accordingly, the more capable pupils may be able to go on with work in borrowing twice in subtraction of three-place numbers while slow learners continue to work with concrete materials and practice on borrowing once.

To provide work with three-place numbers for all pupils, the teacher may construct examples like those below for slow learners. It will be noted that the borrowing is from tens' place only.

$$\begin{array}{r} 394 \\ -127 \\ \hline \end{array}$$

$$\begin{array}{r} 860 \\ -346 \\ \hline \end{array}$$

$$\begin{array}{r} 873 \\ -49 \\ \hline \end{array}$$

GRADE 4. MULTIPLICATION OF THREE-PLACE NUMBERS BY A ONE-PLACE NUMBER. Slow learners must sometimes record many steps which other pupils can do mentally. When the carrying step in multiplication proves unusually difficult, the pupil may be shown a systematic way to record each multiplication fact needed and the carrying step. In each case, the number carried is circled.

$$\begin{array}{r} 348 \\ \times 9 \\ \hline 3132 \end{array}$$

$$\begin{array}{r} 8 \\ \times 9 \\ \hline \textcircled{7}2 \end{array}$$

$$\begin{array}{r} 4 \\ \times 9 \\ \hline 36 \\ +7 \\ \hline \textcircled{4}3 \end{array}$$

$$\begin{array}{r} 3 \\ \times 9 \\ \hline 27 \\ +4 \\ \hline 31 \end{array}$$

In examples such as the above, even the slow learner should understand that the multiplicand means $300 + 40 + 8$. Slow learners who have not mastered work with two-place multiplicands should continue to practice on this phase of the unit before going very far with three-place multiplicands.

GRADE 5. ADDITION AND SUBTRACTION OF MIXED NUMBERS. Pupils of average and above-average ability cover the following variants of this topic in grade five: like fractions as in (a), related fractions as in (b), and unrelated fractions as in (c).

$$\begin{array}{r} \text{(a)} \\ 3\frac{7}{8} \\ +4\frac{5}{8} \\ \hline \end{array}$$

$$\begin{array}{r} \text{(b)} \\ 12\frac{1}{4} \\ -6\frac{1}{4} \\ \hline \end{array}$$

$$\begin{array}{r} \text{(c)} \\ 5\frac{3}{4} \\ +4\frac{1}{4} \\ \hline \end{array}$$

The teacher will need to make available for slow learners individual kits of fractional parts so these pupils may find solutions by the manipulation of parts as long as there is need for them. Attention must be focused upon carrying and borrowing when the fractions are like fractions.

Some slow learners may progress in Grade 5 to work with related fractions as in example (b) while for others the work must be limited to like fractions. The guidance-minded teacher realizes that successful performance on limited aspects of the entire unit is preferable to attempting to move too rapidly to the more complex steps.

GRADE 6. WORK WITH TWO-PLACE DIVISORS WHEN THE TRIAL QUOTIENT IS NOT THE TRUE QUOTIENT. Some slow learners can find correct quotients in examples like the one below if they are guided to show their multiplication steps as side work.

87 R ¹¹	Side work steps		
57 $\overline{)4970}$	57	57	57
456	$\times 9$	$\times 8$	$\times 7$
410	513	456	399
399			
11			

Guidance in Arithmetic for All Pupils

In the preceding discussion there has been no intention of implying that the so-called *average* pupil requires little or no guidance. Quite the contrary is the case. There are adjustments which must be made for all pupils according to their developmental needs. However, the differences in characteristics and needs of the highly gifted at one extreme and the very slow learner at the other occur as a series of continuous gradations. Therefore, many of the activities and procedures described for two special groups in arithmetic are applicable to a greater or lesser degree for pupils at various levels between the extremes.

The teacher occupies the key position in providing the day-by-day guidance necessary to insure each pupil's growth in arithmetic commensurate with his ability. Space limitations will permit merely the listing of some of the necessary competencies of the

teacher as they relate to the field of arithmetic. The following are essential:

1. Familiarity with all the important outcomes of arithmetic, including (a) basic concepts and vocabulary, (b) fundamental principles and relationships, (c) computational procedures, (d) basic facts and knowledge, (e) social and economic information, and (f) problem-solving ability.

2. Knowledge of the learning sequence for the presentation of each of the above aspects of arithmetic.

3. Competence in the selection of appropriate instructional materials from texts, reference books, multi-sensory aids, and community resources.

4. Ability to motivate pupils to put forth the effort needed to reach goals determined in pupil-teacher planning sessions.

5. Skill in the organization of the class into flexible groups as a practical way of meeting individual needs.

6. Skill in the utilization of a variety of procedures for determining each pupil's developmental level in each of the sequential learnings. This prerequisite for successful guidance of all children is expanded in the next two sections.

The Necessity for Integrating Evaluation and Instruction. The discussion on pages 155-56 focused attention upon the long range values of a comprehensive system-wide testing program. However, for the effective direction of the daily learning experiences of children in arithmetic, the teacher must select and administer other types of tests. These may be inventory, diagnostic, readiness, or progress tests, depending upon the purpose to be served. Many present-day textbooks contain satisfactory tests for these purposes.

Inventory tests administered at the beginning of instruction in a grade are often necessary to determine the extent to which the arithmetic learnings of previous grades have been mastered by individual pupils. The scope of the material inventoried will vary according to the grade level, but care should be taken to appraise all aspects of arithmetic, including concepts and meanings, basic facts, computational skill and competence in problem solving.

Mathematically talented pupils will often make perfect or

nearly perfect scores on an inventory test. To subject these pupils to unnecessary redevelopment and practice is to bore them and stifle their interest in mathematics. Accordingly, while reteaching is provided for pupils who require it, superior pupils must be given enrichment activities of the type described on page 167.

When an inventory test covering computational skills has been administered and the results analyzed it will often be found that some pupils need redevelopment of one or more processes while others need assistance on only the more difficult aspects of a single process. Therefore it is advantageous to give a diagnostic test to locate more specifically the shortages in a more restricted area than that included in the inventory test.

The value of both types of tests will be enhanced if teachers make certain that pupils have a positive attitude toward them. Pupils should realize that a diagnostic test reveals strengths as well as weaknesses and that its purpose is to indicate the exact points at which instruction must begin. When pupils accept this point of view and become fully cooperative, they develop the habit of self-diagnosis.

It is often desirable to reproduce a diagnostic test so that pupils may be asked to do all of their work on the test paper. A generous amount of space should be provided beside each example for side work. Pupils should be directed to show all of their figuring and ways of thinking in this side work space. Far too often pupils place on scratch paper work which would give teachers insight into their exact needs. While any type of test is being worked, the teacher might well observe pupils and record significant behavior on an appropriate form prepared in advance.

A fourth grade teacher who gave a diagnostic test on the 100 addition facts noted that some children obtained sums by using immature procedures such as these:

Partial counting, counting on fingers or counting marks

Thinking of doubles as helpers for near-doubles, as in $6 + 6 = 12$, so $6 + 7 = 13$

Making a ten first, as in $9 + 7 = 10 + 6$ or 16

The teacher discovered also that some pupils did not realize that two addends may be reversed without changing the sum and some did not understand the generalizations that apply to adding

0, adding 1, or adding 2 to any number. A lesson was planned to develop these basic understandings.

Some of the above ways of obtaining answers represented more mature ways of working than others. For pupils who consistently used counting, the teacher first provided assistance in *thinking* answers by relating unlearned facts to learned facts. The teacher motivated pupils who had achieved this level to put forth the effort necessary to gain automatic response. Games and short periods of rapid mental computation were used as incentives. Sometimes pupils are not fully aware of the necessity for making absolute mastery the goal for factual learning. This must be emphasized.

A fifth grade teacher who was planning to introduce division with 2-place divisors gave a readiness test to locate pupils who might lack the foundation needed for success with the new unit. Some redevelopment of previous work was necessary because the test indicated shortages for some pupils in these learnings:

Inadequate mastery of the even division facts with divisors 6, 7, 8, and 9

Lack of understanding of the relation of uneven division facts to corresponding even division facts

Incomplete mastery of computational procedures involved in examples such as these:

$$\begin{array}{r} 29 \\ 3 \overline{)87} \end{array} \quad \begin{array}{r} 203 \\ 8 \overline{)1624} \end{array} \quad \begin{array}{r} 417 \\ 9 \overline{)3753} \end{array} \quad \begin{array}{r} 113 \text{ R}^4 \\ 7 \overline{)796} \end{array}$$

Because four pupils in this class were extremely slow learners, the teacher decided to omit the presentation of work with 2-place divisors for these pupils. They were assisted in further work with 1-place divisors when the remainder of the class acquired the prerequisite learnings and proceeded with more involved steps in division.

This particular class contained two very capable pupils whom the teacher encouraged to work at their own rate in the unit. After a limited amount of introductory work, these pupils completed successfully many examples in which the trial quotient was not the true quotient—a step for which the majority of the class did not have the readiness.

Providing Assistance by Individual Interviews. Paper-and-pencil tests as a means of determining pupils' status upon entrance to a new grade or at the beginning of a new unit of work must be supplemented by oral tests, group discussion and individual conferences. These techniques should also be used to uncover faulty thought procedures, misconceptions or purely mechanical methods of working.

When Fred was asked to think aloud as he worked the example below, his teacher discovered that he was using a mechanical procedure with no basis in understanding.

$$\begin{array}{r} 6\frac{3}{8} = 5\frac{11}{8} \\ -1\frac{7}{8} = 1\frac{1}{8} \\ \hline 4\frac{4}{8} = 4\frac{1}{2} \end{array}$$

To transform $6\frac{3}{8}$ to $5\frac{11}{8}$ Fred said, "I take 1 away from six and write 5. Then I look at $\frac{3}{8}$ and add the 3 to the 8 and get 11. I write 11 over 8 to make $\frac{11}{8}$ so that I can subtract $\frac{7}{8}$."

Fred's mechanical trick worked when he remembered to reduce the whole number by 1, but often he forgot this step because it was meaningless to him. He did not realize that he was using 1 whole (subtracted from 6) and transforming it to $\frac{8}{8}$, then combining $\frac{8}{8}$ and $\frac{3}{8}$ to get $\frac{11}{8}$. Fred had no concept that $5\frac{11}{8}$ was equal in value to $6\frac{3}{8}$ although different in form. When paper plates and their fractional parts were used to demonstrate the transformation, Fred understood why his trick worked.

In helping Fred and others who lacked understanding of this process, the teacher related the subtraction of fractions to the same process with whole numbers and with denominate numbers. The following transformations of the minuend to facilitate subtraction were explained:

$$\begin{array}{r} 72 = 7 \text{ tens } 2 \text{ ones} = 6 \text{ tens } 12 \text{ ones} \\ -39 = 3 \text{ tens } 9 \text{ ones} = 3 \text{ tens } 9 \text{ ones} \\ \hline \end{array}$$

$$\begin{array}{r} 7 \text{ ft. } 2 \text{ in.} = 6 \text{ ft. } 14 \text{ in.} \\ -3 \text{ ft. } 9 \text{ in.} = 3 \text{ ft. } 9 \text{ in.} \\ \hline \end{array}$$

Sally, in the fifth grade, was interviewed with regard to her method of reducing fractions to lowest terms. She was usually

successful in reducing fractions such as $\frac{6}{10}$ or $\frac{12}{18}$, in which she could use an even number for a divisor of both terms. She failed to reduce fractions such as $\frac{6}{9}$ or $\frac{12}{15}$ because she had the mistaken idea that only even numbers could serve as divisors.

The teacher found that Sally lacked other basic understandings for fractions. She was given the opportunity of working with fractional parts to discover that reducing $\frac{6}{9}$ to $\frac{2}{3}$ may mean replacing 6 parts of a disk divided into 9 equal parts by 2 parts of the same sized disk divided into 3 equal parts.

Fred and Sally are typical of pupils with average ability whose achievement in arithmetic is below average because of misunderstandings and partial learnings. These must be discovered by the teacher and remedied. To enjoy arithmetic and be successful in it, pupils must expect to understand the reasons why procedures give correct answers. They must have the courage to ask questions when they do not see sense in the work. The teacher can often rebuild confidence and change a pupil's negative attitude toward arithmetic to a positive one by his sympathetic approach during an individual conference.

SUMMARY

In this chapter we have identified certain guidance concepts that have particular significance for the elementary school. We have outlined the essential features of a recommended program with emphasis upon a cumulative record system and a comprehensive testing program. A guidance program in action with implications for one subject-matter area, arithmetic, was described in the last part of the chapter.

For the successful integration of guidance with classroom teaching in arithmetic, there must be assistance for teachers from skilled counselors, supervisors, and administrators. This group of specialists must assume responsibility for making available to teachers modern textbooks and manuals, diagnostic and achievement tests, and learning aids. Teachers must develop effective procedures for the utilization of these materials. Assistance in working with parents and in handling unusually difficult cases of emotional and social maladjustment must also be provided.

It is the firm conviction of many educators and parents that if facilities were available to enable schools to adjust the learning experiences of each pupil to his specific needs and abilities, and if each pupil's activities were motivated in harmony with his interests, there would be a sharp decline in the number of pupils who experience personality and emotional difficulties.

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Mental Hygiene and Arithmetic

JOHN W. DICKEY and EDNA B. TAYLOR

Well, three or four months run along, and . . . I had been to school most all the time and could spell and read and write just a little, and could say the multiplication table up to six times seven is thirty-five, and I don't reckon I could ever get any further than that if I was to live forever. I don't take no stock in mathematics, anyway.

—*The Adventures of Huckleberry Finn*, by Mark Twain

A FUNCTIONAL PROGRAM in arithmetic rests upon a modern philosophy (11) (17) and a modern psychology of education (33) (37) (49). The program which has developed over the past 25 years seems adapted to the needs of society as well as to the capabilities of the learner (8). The total personality of the learner has become the concern of modern education. He is asked to understand mathematical meanings of arithmetic as well as the uses to which his learnings will be put in social living. His habits of knowledge and skill are very important, as are those habits of

his emotional life. Chief among the habits of emotion are those of attitudes (9). Both common observation and careful research agree that they play a major role in learning arithmetic.

While the stated subject of this chapter is the mental hygiene of teaching and learning arithmetic, the major concern reduces itself to the effect of attitudes upon a successful program in arithmetic. It is of interest to note that no single case of severe emotional maladjustment tied to arithmetic has come to the attention of the writers. A recent survey made by one of the authors of this chapter finds complete agreement as to the central role of attitudes in connection with the learning of arithmetic, in the judgment of a number of competent educators. Three mental hygienists, two clinical psychologists in private practice, two public school psychologists, and two public school administrators working in the New Jersey metropolitan area were interviewed. Not a single case of severe emotional disturbance tied to arithmetic had come to their attention; however, problems involving attitudes related to arithmetic were frequently observed by them. These educators were in full agreement as to the importance of the relation of attitudes to the successful learning and teaching of arithmetic.

ATTITUDES AND ARITHMETIC LEARNED TOGETHER

A number of studies have been made on attitudes as they are related to learning arithmetic. Brown (7:61-62) in 1933 obtained 22 statements from graduate students who were taking a first course in educational statistics where arithmetic was used quite extensively. The statements were made in response to the question, "Do you find the going pretty tough?" Twenty out of the 22 students experienced difficulty. Among the responses were the following: "I might if I knew what it was all about," "My trouble is all with my arithmetic," "Believe me, if I get a mark in this course I'll never take another," and "Boy, I know the words, but I can't carry the tune." From the writer's own teaching experiences in this area, he can appreciate the difficulties of applying arithmetic in educational statistics and the relative ease with which the older attitudes toward mathematics seem to recur as fixed habits.

The most elaborate studies of the attitudes associated with learning arithmetic have been made by Dutton working at the University of California at Los Angeles. In 1951 he surveyed the attitudes of prospective teachers toward arithmetic (18), and received 211 responses, 74 percent reflecting unfavorable attitudes. For a summary of this investigation, turn to Chapter 13, page 300.

A second study on attitudes (19), also by Dutton, was made on a Junior High School group of 226 boys and 233 girls in Los Angeles, California. This sample of 459 was drawn from a total school population of 2200 pupils, representing three different income levels and a wide variation in language and cultural background. Of the 22 items on an attitude scale, those listed near the middle received the highest percentage of responses. The items with the greatest frequency were, "I enjoy doing problems when I know how to work them well" (87 percent), and "Arithmetic is as important as any other subject" (83 percent). Others, expressing either a favorable or an unfavorable attitude, had fewer responses. When these 459 pupils were asked why they liked or disliked arithmetic, the favorite reply (132 times) was, "Need arithmetic in life, future needs, practical applications." Next in frequency of response were, "Arithmetic is interesting" (66 times) and "Arithmetic is fun" (64 times). The statement most frequently checked to indicate a dislike was, "Don't understand arithmetic" (57 times). Others were, "Arithmetic is too difficult and complicated" (29 times), "Boring, same things over and over, can't memorize" (26 times), and so on. Dutton concludes this study of junior high school pupils with several findings. The chief finding indicates that pupil dislike for arithmetic centered in lack of understanding, difficulty in working problems and poor achievement. On the more constructive side, good teaching and understanding and an appreciation of the use of arithmetic in life seem basic to the formation of favorable attitudes toward this subject.

A third study by Dutton of attitudes associated with arithmetic (20) reports the construction of an attitude scale using the principles employed earlier by Thurstone and Chave (50). This scale was administered to 289 students in education courses. The main conclusions resulting from this survey indicate that while

feelings are developed toward arithmetic in all grades, the most crucial are grades 3 through 6 and the junior high school; that students liked arithmetic when they understood it, when they thought it was practical and definite and provided them with a challenge; they disliked it when it was too difficult, or when it made them feel insecure, or when it was poorly taught. It is also interesting that any part of arithmetic can be liked by some people and be disliked by others.

Two additional studies of considerable interest have been made concerning this problem of attitudes associated with arithmetic. Chase (10) surveyed the preferences of 13,483 fifth-grade children in New England for their school subjects. More than one-fifth preferred arithmetic over all other subjects. Only reading, with a very slight fraction of a percent preference over arithmetic, was more popular. While no reasons are given for these preferences, it is interesting to note that among the 543 teachers of these children, arithmetic was the favorite subject. A second, much less extensive, survey of the general liking for arithmetic was made by a college student of one of the writers. He interviewed more than 50 elementary school pupils drawn from a rather wide area in metropolitan New Jersey, and found the full range of likes and dislikes. Reasons for these attitudes were the customary ones, turning on such points as an understanding of the work, the quality of teaching, and the many varied teacher-pupil relationships.

Numerous judgments of a more subjective nature have been made in recent years about the prevalence of unfavorable attitudes concerning arithmetic and the necessity for doing something about this whole problem. Brown is concerned about the welfare of our country (6). Rogers (44) suggests numerous ways to build the right attitudes, such as avoiding meaningless content and drill and establishing readiness in the learner for the material. She discusses the need for the teachers to understand and enjoy both the children and the arithmetic. She feels that in the interest of mental health this problem needs our attention. Cooke (14) suggests that if learnings are geared to the developmental rate of the learner, both understandings and right attitudes are apt to follow. Johnson (30) is most appreciative of the problem, and suggests a number of attitudes that he would like to have

developed in the mathematics classroom. Two examples cited by him are those attitudes associated with the feeling that the subject is important in our daily lives and in society, and those associated with the desire to learn and to gain an insight into the structure of mathematics. Stone (48) believes that elementary school pupils are handed over to secondary schools "with an abiding distrust, even a deeply ingrained fear, of mathematics." He concludes that "the explanation of this phenomenon must lie in the manner of its teaching." Clark (12) lists eight "Issues in Teaching Arithmetic"; the first issue listed indicates a concern with the problem of attitudes. Other writers have also expressed themselves, but the above are representative of the interest in improvement of attitudes learned along with arithmetic.

Mario Salvadori, professor of civil engineering at Columbia University, presented a challenging speech on "The Universal Fear of Mathematics" at the 35th Annual Meeting (1957) of the National Council of Teachers of Mathematics. He observed that young children enjoy counting, but as the authoritative teacher presents this material deductively the freedom for discovery is thwarted, the pupil's creative powers are gradually lost, and there is very little opportunity for him to be free to make discoveries. Puzzles in mathematics are abstract and provide a challenge for the brighter pupils, but practical applications of mathematics provide the challenge for average and below average pupils. Once more the quality of teaching, the understanding of the pupil, and the understanding of arithmetic seem to be ever-recurring themes associated with development of favorable attitudes toward arithmetic.

The studies summarized here represent some of the thinking of educators on this important problem of the mental hygiene of teaching and learning arithmetic. The major conclusions seem rather clear. Attitudes must be considered in a successful arithmetic program. Attitudes are learned, and they can be found on any level of maturity. It is important, also, that the teachers understand their pupils as well as the arithmetic, and use superior methods to promote fuller arithmetic meanings for the learners. These factors contribute substantially to acquisition by the pupils of those attitudes which are tied to arithmetic learnings.

The remainder of this chapter concerns itself with some of the

major factors in an educational program which would help to build favorable attitudes as a part of learning arithmetic. It deals with basic principles of mental hygiene which can be applied to the teaching and learning of arithmetic in the present-day program of making arithmetic meaningful, and offers a fuller discussion of the larger Gestalt in which a child lives and learns. It is hoped that this discussion will enlarge the conception of the total function of teaching, and that it will promote better practices in the teaching and learning of arithmetic.

BASIC PRINCIPLES OF MENTAL HYGIENE

Basic principles of mental hygiene as applied to the teaching and learning of arithmetic are the same principles which should underlie the entire process of education.

Mental hygiene comprises not only the science but also the art of preserving soundness (wholeness) of mind. Organic brain disorders—whether from injury, infection, tumor, or other less-clearly-defined causes—are the province of the physician. So also are those “functional” derangements, including severe neuroses and the psychoses, which incapacitate the individual for normal living and may even require that he be hospitalized. But maladjustments of lesser degree which permit the individual to keep going after a fashion, yet make him unreasonable, unpredictable, unhappy, unmanageable, and generally hard-to-live-with—these impinge on us all, and are of special concern to the educator.

Malformations of personality begin early in life in the home and are still in the formative stage when the child enters school. Rarely are they full-blown or irreversible. Through intelligent appraisal, sympathy, patience, and effort something can usually be done to modify the deformity, reconstruct the child's attitude to a degree, and salvage one more person for constructive living. Complete, detailed recipes for good mental health are hard to formulate, though we may list many essential ingredients. We are quick to recognize the finished product. We sense it in the behavior of the well-balanced individual who takes the ups and downs of daily living in stride, who can be counted upon to carry his share of the load cheerfully, to be fair and objective in his dealings with others, and

to continue to grow in constructive achievement, range of interests, and depth of understanding. Although character and personality may defy precise analysis, it is instructive to examine the soil in which they grow, and the processes whereby they are cultivated.

Conditions Essential for Attaining Mental Health

Despite marked differences in temperament and capabilities, children with sound mental health share a common core of experience. From the important persons in their world they have received and enjoyed respect for their individuality, consistency and a sense of security, freedom to grow, and guidance and encouragement toward maturity.

Individuality. Whatever his shortcomings, however limited his endowment, every individual is entitled to recognition and acceptance, without reservation, as a unique entity. This is the cornerstone in the edifice of human rights. The process of education should protect and preserve the pupil's individuality, even as it explores and develops his potential, to the highest degree compatible with the rights of other individuals and the welfare of society as a whole.

Consistency and Security. We live in a universe of law and order, characterized by unceasing change. The first important step toward understanding this concept is taken easily by the child when adults around him behave consistently and predictably. Overstrict discipline is no longer the ideal of teachers, and rightly so. Yet even strict discipline, if steadily maintained, helps build up the idea of system and order, and does far more to establish a sense of security than does arbitrary indulgence. Affection and acceptance are basic to sound development, but they cannot function effectively in chaos—in situations where the child is kept constantly guessing what's going to happen next.

Freedom to Grow. Freedom to grow is a basic necessity and should require no emphasis. The teacher's vocation fosters it in the finest sense of the word. Sometimes, however, in the interest of efficiency, he may be too eager to hasten the process—to complete the blueprint as he interprets its design. The opportunity

to make mistakes and to learn from them is, of course, an integral part of growing. Progress is likely to be steadier, and ultimately more rapid, when emphasis is placed upon sound thinking, a more effective approach, rather than upon the failure or misdoing of the learner.

Guidance Toward Maturity. Making maturity attractive and helping the child advance steadily toward it without too many set-backs are the great educational goals. Many teachers would agree that in practice true maturity is increasingly approached, rather than fully attained. At least four cardinal qualities seem essential to maturity: stability, adaptability, reliability, and integrity. Any one of these traits can be elaborated and its development traced, insofar as it relates to the educational process. The analysis which follows will be brief, discussion limited to fundamentals, and application confined to the teaching and learning of arithmetic.

Emotional stability has many factors. Its seed is planted in the soil of respect and affection; it flourishes in an atmosphere of objectivity; it is nurtured by constructive achievement, and it flowers as a well-developed sense of proportion. Given a favorable setting (reciprocal respect and affection between teacher and pupil), scarcely any other subject in the curriculum offers as rich an opportunity as arithmetic to promote stability.

Unlike the humanities, where opinion and interpretation necessarily form an integral part of the subject matter, arithmetic is supremely objective, predictable, and precise. There are no qualitative shadings to arouse uncertainty. With rare exceptions there is only one correct answer. Having worked his way to it, the learner is on solid ground. Missing it, he can retrace his steps in orderly fashion and discover, often without help, where he went off course. Once he has mastered the multiplication table, rote memorizing decreases in importance while the process of logical association becomes an increasingly valuable tool. He *knows* what he has achieved, can assess his progress quantitatively, and, as a result, enjoys a sense of satisfaction rarely equaled. Certainly his ability to weigh values, balance judgments, and test his thinking soundly should grow stronger and finer in the course of learning. Finally, if both content and presentation can be so geared to the

student that he understands each lesson in arithmetic, finds it practical, definite, and challenging, his floundering should be infrequent, his frustrations brief. Successful achievement builds confidence, without which emotional stability can scarcely be maintained.

Adaptability is like a scooter—swift, simple, compact, and unpretentious. Properly employed it can take us where we want to go, around obstacles and along busy thoroughfares, without waste motion or loss of balance. The study of arithmetic may streamline the learner's progress from random, trial-and-error activity to inductive reasoning. Gradually, as the pupil increases his understanding of the structure of arithmetic, he becomes more and more adept at distinguishing essentials from non-essentials, and breaking down larger categories into their logical components. Solving a practical problem requires the application of mathematical principles and procedures. This calls into play both adaptability and ingenuity. From an accumulation of such instructive experiences the firm residue of resourcefulness is formed.

Reliability, the third essential of maturity, was perhaps over-emphasized in the old-time school, but its near-absence in many high school graduates today is commonly criticized. Yet the social setting of the classroom provides, as never before, opportunity to establish a sense of responsibility. Individual assignments can serve as a stimulus; informal class discussions put it to the test. For a child's peers will criticize him with a frankness—even ruthlessness—which an adult would hesitate to use for fear it might be too damaging. Surprisingly enough, the target of such criticism often accepts it, and teachers know well how such lessons can speed up the learning process. Gradually the youngster may come to realize that it pays, in the long run, to stick with a task until it is finished; to be there, whether he feels in the mood or not; to do his best with or without plaudits.

Integrity might be called the final fruit of maturity. It can develop fully only as other fundamental lessons are learned and practiced thoroughly. There is no need to enlarge upon the process here, for the teacher of mathematics understands the full significance of the term most exactly—an integer being "a whole number, a complete entity, undivided and unbroken." Patience,

persistence, and constructive self-criticism are all involved in learning arithmetic. Their application here is direct; the fact that such effort *pays off* is easily understood by the pupil because of the exactness and precision of the subject matter.

Important Principles

"This is all very fine," you say, "but where does it lead? What am I to do about the day-to-day infractions, the individual deviations not of just one but of some twenty or forty unique little 'entities'? These incidents may not be serious, but they can certainly disrupt classroom discipline, foster the wrong kind of learning, and whittle down my patience."

This question is practical, and it deserves a realistic answer. Two basic principles can serve as pointers. The teacher who subjects them to critical analysis, puts them into practice, and then weighs the results carefully will probably agree with the following: First, all human behavior—however inadequate, reprehensible, or bizarre—makes sense if we view it as an attempt to adjust; second, human beings and human welfare are of prime importance and concern for them should take precedence over preoccupation with things. If we accept these hypotheses it follows, as a corollary, that human time and energy are precious, and should therefore be conserved.

Our first principle parallels the law of inertia in a human sort of way which is not entirely predictable. The individual tends to do and to continue doing what he has found to be rewarding. Recognition in some form—and its guises are manifold—ranks very high in the scale of values.

Attention we crave,
from cradle to grave.

If only acceptable behavior is accorded such recognition, in time the unacceptable should wither from neglect. Happily for the teacher, educational progress is often sustained by applying this principle constructively. But what of the negative reaction—doing nothing, giving up, withdrawing into daydreams? Or—at the opposite extreme—flouting authority so flagrantly as to invite

punishment? What kinds of adjustment are these, and how are they, in turn, to be corrected?

We must sometimes delve for the answers, but three pertinent questions can help:

1. What pressures and tensions does this specific behavior seek to relieve?

2. What immediate pleasure does it bring, and/or what secondary gain does it achieve?

3. Toward what ultimate goal, if any, is it directed?

A dozen types of reaction will come to mind at this point, ranging all the way from restless wriggling or momentary inattention to stubbornly resisting every effort on the part of the teacher to get at the child's basic problems and help him solve them.

In almost every instance strong emotions are shifting the gears, throwing into high all the emergency equipment designed to facilitate swift movement. For it is chiefly by means of a series of quick, successful adaptations carried out by the neuromuscular organs that our human species has managed to survive. This original function of the sympathetic nervous system persists unchanged. Since energy can neither be created nor destroyed, today—as always—emotion must ultimately be discharged in motion, i.e., muscular movement, if a healthy balance is to be maintained. A major goal in the educational process is to utilize this potential power by providing suitable channels for its flow and guiding the child to find them.

High on the list of tension-producing emotions stand fear, frustration, and resentment. In the classroom, dread of ridicule may outweigh fear of actual failure, yet lead to failure by stifling learning. Frustration pyramids as failures multiply. If the principle underlying today's problems in arithmetic is poorly understood, the answers may be not only technically incorrect but also quite devoid of meaning. Unchecked, this process becomes malignant, teaching the child to expect failure and to accept frustration. The natural reaction of growing resentment only adds to his burden by depleting the energy available for objective thinking. Ultimately something has to give; the harried youngster throws in the towel and starts to misbehave—drawing pictures, whistling under his breath, launching paper planes. What began, perhaps, as

random activity for activity's sake is presently noticed by his classmates and interpreted as deliberate misdoing. Their surprise and admiration provide the incentive he needs; so he persists until the teacher stops and reminds him to get back to work.

Right at this point an inspiration seizes him. With the wounded air of the righteous unjustly accused, he dramatizes his difficulty, and succeeds for a time in cornering the instructor's undivided attention as the latter attempts to explain. When this foray is over he understands the problem little better than before, having paid it scant heed. He is much too busy with the delight of discovery—how to turn on attention at will, like a faucet, at the same time postponing an unwelcome chore. The pupil in question will scarcely discern or deliberately select an "ultimate goal" for his behavior. The immediate pleasure it brings is reason enough for trying again. But if he stumbles on the stratagem of the surprise attack, if he uses ingenuity in varying his tactics, and if the initial reward was sufficiently sweet, he will be in a fair way to win the wrench-throwing championship in his particular class. Becoming a problem was not his original intent. All he wanted was to feel more comfortable by escaping from excessive pressure. This disturber of the peace was simply trying to adjust, finding pleasure and secondary gain in the process, thus illustrating clearly our first basic principle.

To the second postulate, the prime importance of persons over things, most of us give lip service. Prone to forget himself as a person, the teacher may gradually become the servant of routine rather than its master. To correct this he should remind himself from time to time of the following: mastering the principles of sound thinking has greater permanent value than merely learning subject matter; the habit of investigating fosters growth and should be encouraged, even though it may threaten, at times, to disrupt a smooth routine; learning to manage one's emotions—to realize their strength, respect their potential, and apply this knowledge to constructive achievement—is probably the greatest single factor contributing to mental hygiene; once the child enters school, the character and attitudes of his teachers will probably do more to help or hinder healthy growth than any other single factor outside the home itself.

Stages and Setbacks in Development

As the child grows he is dealing constantly with three different forces: his own basic needs; the conditions imposed by his physical environment; and the dictates of conscience—that growing body of standards, ideals, and attitudes inculcated in him from birth, which are characteristic of the particular culture in which he develops. The child is expected increasingly to reconcile, to control to a degree, and in some measure to understand these three forces. This is a large order, but if the process goes well he grows into a civilized adult whose conscience functions as a strong inner authority, sustaining and restraining his behavior far more effectively than external force.

Fortunately, the child himself usually sends out his own particular radar signals of poor adjustment in the making. These include:

1. Unusual emotional reactions
2. Excessive inattention or apathy
3. Chronic negativism or aggression
4. Marked disparity between intellectual capacity and scholastic achievement.

Though not a complete catalog, this includes most of the commoner problems.

1. These flare-ups are either disproportionate or inappropriate to the stimulus apparently provoking them, or both. "Apparently," because there is always "more than meets the eye," the precipitating cause merely touches off a supercharge of tension. The youngster who flies into a rage when another pupil accidentally knocks a pencil off his desk and the girl who freezes when called upon to recite may both suffer from too much restriction at home.

2. Excessive inattention or apathy, when no actual impairment can be demonstrated, is often the trademark of a child chronically insecure and discouraged. By not becoming involved he attempts to barricade himself against further injury. Carried to extreme, however, such withdrawal is one sign of a very serious disturbance, early childhood schizophrenia. Whatever the child's individual problem may be, fear of further failure paralyzes each attempt to solve it. This is a malignant, progressive process. To check it, expert professional help is almost indispensable.

3. Chronic negativism and aggression are the reverse and obverse of the same coin—reaction to frustration. Arbitrary, unnecessary restraint by adults is justly resented. Attempting to remove frustration completely, however, or even to reduce it to an unrealistic minimum, is not only impossible but highly undesirable. Maturity is synonymous with knowing how to pick up and go on after failure, how to endure illness, weather financial loss, and carry patiently the burden of anxiety.

4. The gifted pupil whose grades are below passing is quickly recognized as having emotional problems. His very intelligence, however, may make him adroit at resisting help because he has, unconsciously, a vested interest in failing. One example will suffice. Two overactive, socially ambitious parents have always taken their son for granted, and shown scant interest in any of his activities unless he got into trouble, in which case they would begin to fuss. This sort of attention is a poor substitute for the love he misses, but he has learned the hard way that half a loaf is better than no bread.

Aids to Maintaining Mental Health

Few persons will dispute the hypothesis that an individual's unsolved emotional problems are likely to spill over into less private and personal areas of his life. This is especially true of the disturbed teacher, whose inner turmoil will almost certainly distort his view of the pupil's difficulties. If we adopt the crudest criterion, and evaluate him solely as an educational machine (to be maintained at top efficiency in the interest of increased production) the teacher's emotional adjustment must still have high priority. Equally with the children he teaches, he is entitled to be a person in his own right, to balance and expand his life, and to continue to grow. The routes to professional growth and advancement are well charted. Still to be explored are other important areas such as religion, home, and recreation.

Religion is pertinent to this discussion for a very important reason. Many well-balanced persons cherish faith in a Supreme Being and adhere to ideals which they apply in daily living. Their beliefs can stand intelligent scrutiny, are free from supersition, and have become much more than parental hand-me-downs. The quality of

character achieved by the believer demonstrates the lasting value to him of his beliefs. Medicine sheds an interesting sidelight on this subject. A sizeable number of psychiatrists have recorded the observation that very few patients who consult them can qualify as being spiritually well-nourished and robust. Deprivation and disturbance in this area are almost routine findings.

Making his own home also has important implications for the teacher. The simplest one-room apartment becomes a haven after a hard day's work, and it makes possible a flexibility of schedule and freedom for hospitality rarely enjoyed by the teacher who boards. When he is young, single, and slender of salary he may decide to continue living with his parents, paying only nominal board and rent, but this arrangement should be temporary. Though self-supporting and professionally competent, the son or daughter who lingers under the same roof remains a child in his parents' eyes. Marriage is likely to be postponed, maturity almost certainly curtailed over a period of years. One further word of caution. The decision to share living quarters with colleagues should be made deliberately, after the persons involved are thoroughly acquainted. Professional tensions being what they are, maximal congeniality and minimal need for adjustment will spare everyone concerned, and will pay rich dividends in lasting friendship.

Recreation worthy of the name is freely chosen, devoid of ulterior aim, and tailored to individual needs. There is no disputing tastes, but teachers frequently find that the greatest gains come from interests that are active and creative. Sports suitable for the adult whose work is taxing and sedentary may, however, be non-existent in the very areas where athletics for youngsters abound. The most absorbing hobby, too, can lose its appeal if it is pursued in solitude. These handicaps aside, teachers as a group are notable for unusual and diversified leisure-time pursuits. Among the most frequent and systematic of travelers, they are likely to go exploring off the beaten tourist track, and to discover, in the process, the stimulating pleasure that comes from association with persons of widely differing occupations and diverse opinions. Such experiences can be a real tonic to the teacher, whose work requires him to spend so much of his time in contact with less mature minds. It may even help to protect him against the great occupational disease, creeping authoritarianism.

To summarize briefly, mental hygiene begins with sound emotional adjustments in early life. The child who is respected as an individual, treated consistently, and given security and freedom to develop has an excellent chance of reaching maturity in sound mental health. Growth is a process of continuous adjustment to the demands made by basic needs, environmental conditions, and conscience. Conflicts must be resolved if a workable compromise is to be reached, but in the struggle severe anxiety may be aroused and repressed before a satisfactory solution can be found. Important signs of maladjustment include disproportionate or inappropriate emotional reactions, excessive inattention or apathy, chronic negativism or aggression, and marked disparity between intellectual capacity and scholastic achievement. Since the teacher's unsolved emotional problems can affect his pupils adversely, his continuing growth and health of mind are vital factors in the educational process.

CONTENT AND METHOD: OLD AND NEW

The present program in arithmetic attempts to be functional. It not only aims to serve the individual in his social living, but also aims to contribute to his personality development through a fuller understanding of the nature of arithmetic. Both of these aspects are reflected in the changes in content and methods of today's program (21) (52) and are quite at variance with the program around the turn of the century. A look at the textbooks in use somewhat before 1900 will show a contrast with those now in the classrooms. The older books were quite small in length and width, and the print was smaller than any standards of health would condone today. Only a few black and white illustrations were included, usually in that portion of the book concerned with measurement. Quite large examples in whole numbers were in evidence. These earlier textbooks had tables of apothecaries' and troy weights, tables of cloth and liquid measure expressed in units now almost forgotten, and many other topics now deleted from modern arithmetic texts. It is doubtful if much of this material was functional for children in the way we are trying to present arithmetic

today. Without much practical use, its chief contribution was thought to be in its mental discipline values.

It is somewhat redundant to attempt a detailed picture of modern arithmetic textbooks. Perhaps a few observations will suffice to point out some of the major changes that have occurred over the past century. The format of modern textbooks places them among the most attractive books in the world. The pages are abundantly illustrated in color rich in chroma, which appeals to the young, and for the most part the illustrations are pertinent to development of the material. The content is usually organized around mathematical units of interest which have been placed at levels largely in keeping with the maturity level of the learner. Word problems and numerical examples reflect solid and useful information and develop skills needed in our daily living. A pupil today cannot fail to sense the note of realism in his arithmetic work, and his attitude toward it should be improved as a result of this realism.

When we compare the methods of arithmetic for the two periods, we note a number of major changes. The older methods were largely deductive. Rules and definitions were learned by rote in most cases, then followed by examples and problems to be solved in a manner indicated by these rules. Of course there were good teachers who tried to make the work meaningful, but the textbook gave the teacher little or no help in this effort. (Colburn and those who imitated him were somewhat of an exception.) In today's classroom the picture is materially changed. The laboratory approach, which makes use of concrete materials and many questions asked by both the teacher and the pupils, culminates in the formulation of generalizations which should be quite clear in the thinking of pupil and teacher. Finally, practice lessons are engineered to fix these learnings (46). A variety of modern methods fosters a maturing mathematical sense: the estimation of approximate answers, the checking of answers for reasonableness, the development of habits of analysis with word problems, and the ever-present sense of the reality of the content being discussed. All this should result in more meaningful learnings, and in the development of more favorable attitudes by pupils and teachers alike.

THE EMOTIONAL CLIMATE OF LEARNING

Human relationships are always involved in any educational enterprise, and the teaching and learning of arithmetic is no exception to this rule. The pupil-teacher relationships, the organization and administration of the school including the role of the teacher in it, and the teacher-parent relationships all provide a behavioral environment in which the learner operates. The many forces, attractions, valences, blocks, and the like, in a large measure determine the interests, needs, attitudes and habits acquired by the pupil. A few remarks about these relationships will serve to emphasize the importance of each.

Numerous factors have been shown to enter into these teacher-pupil relationships involving arithmetic. Those attitudes held by the teacher are quite contagious. Kotinsky and Coleman (35) say "teachers' attitudes are strategic," and "without the right ones all else will fail." Ratanakul (43) of Bangkok, Thailand, says,

I began to enjoy learning arithmetic when I was in grade three, because I had a very kind and nice teacher. Later on I hated arithmetic again because I had a terrible teacher once more. The very thing I always remember is that whenever the teacher finished her explanation, she asked the class to raise some questions, and once I did. Instead of getting her helpful answer, she said that I did not understand and made an ugly face at me. I remember that I was very furious at her and since then I hated both the subject and the teacher.

Attitudes formed during such traumatic experiences are driven deep into the personality and are hard to change. In most cases they are formed more slowly. One such case of the slower learning of attitudes occurs when the teacher recognizes the success of the learner. In discussing the principles of learning applied to arithmetic, Hildreth (26) states that "attitudes are tied with success." She believes that teachers should recognize the honest effort of the pupil. Clark (13) expresses the conviction that if good teaching is carried out, the intangibles will take care of themselves. Cooke (14) believes that a positive attitude favorable to arithmetic can be acquired through the meaningful learning of the concepts of arithmetic. These expressions of the varied ways in which a teacher can promote the right attitudes in the give-and-take of the

teaching and learning relationships are but a few of the many such views held by people in education. Other ways have been studied and analyzed (27). Whether these attitudes are acquired by contagion, or through traumatic experiences, or learned more slowly as intangible by-products accompanying good teaching, one cannot gainsay the point that they are learned and that they are intimately tied with both the subject matter and the teacher during these numerous teacher-pupil relationships. The awareness of this psychological fact should motivate teachers to make an honest effort to make functional the basic principles of mental hygiene when teaching arithmetic to these personalities in their most formative years.

There are numerous factors inherent in the organization and administration of the school which can promote desirable attitudes toward the whole field of learning arithmetic. Only a few such factors will be mentioned. The provision of up-to-date textbooks and teaching materials makes a tremendous difference, especially since most teachers in public schools are *textbook teachers*. The spotting of exceptional children and the organization of an enriched program for them will add a tone to the entire school which will be most worthwhile (15) (24) (31) (56). A willingness to use some of the school moneys to provide an abundance of supplementary books (25) (28) which can enrich arithmetic teaching and the willingness to provide small classes will do much to create the best in attitudes for the pupils and the teachers. Then, too, good attitudes result from the solid support of the principal when the teacher is willing to try out one of the newer experimental methods (47) (55), such as the recent use of the adding machine (22) (45).

A third important factor which contributes much to the emotional climate of the arithmetic program is the attitude of the home toward mathematics (42). It is not unusual to hear a parent say, usually in good humor, "I never was any good in arithmetic, and I can't blame Mary for not doing so well in it." To admit to not having read one of the more recent books-of-the-month would be somewhat embarrassing, but to admit to a most modest achievement in arithmetic is rather popular in all too many quarters. In many cases, of course, the parent has done much better

in mathematics than one might conclude from his remarks; in other cases, the expressions are quite valid. The child may not be able to evaluate fully such a remark made by his parents. He may not be able to see back of the facade and catch the truth. What he is more apt to do is to accept it at face value, and this he may experience even before he is old enough to enter school. Here is a pressing problem to be solved, perhaps in parent-teacher meetings or parent-teacher conferences. Doubtless, it will not be an easy problem to solve. Additional conferences which should also contribute to the emotional climate of learning may be those held in connection with the written work in arithmetic, those concerned with the revision of the report card and the objectives of arithmetic which may be a part of it, and those related to the many human relationships occurring during the daily work with the class. Thus the teacher, the principal, the child, and the home all play a part in this learning enterprise involving attitudes.

CONCLUDING STATEMENT

Arithmetic and its methods have changed considerably over the past century. With increased knowledge of and concern for the whole child, the formation of desirable attitudes is a very important part of a functional program in arithmetic. It seems essential that more attention be given to the larger Gestalt which is the child's world. Thus teachers, administrators, children, and their parents have the child's world as a common meeting ground where all can work for a more effective arithmetic program. This joint endeavor should result in better teaching and learning of arithmetic in an emotional climate which engenders favorable attitudes toward arithmetic. It may be that our national survival is contingent upon such total learnings.

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Reading in Arithmetic

PETER L. SPENCER and DAVID H. RUSSELL

READING in arithmetic is very different from conventional reading. The Yearbook Committee asked Dr. Peter L. Spencer and Dr. David H. Russell to write on this subject. Dr. Spencer, in the first part of the chapter, discusses the nature of reading in arithmetic and some of the difficulties involved. Dr. Russell, in the second part, "Arithmetic Power Through Reading," presents a survey of research dealing with reading in problem solving and suggests ways teachers may help pupils develop skill in this area.

F. E. G.

PERSPECTIVE IN MATHEMATICAL READING

PETER L. SPENCER

Reading printed words and developing abilities for reading them effectively have been of major concern throughout the history of American education. As early as 1647 Massachusetts Colony made instruction in printed word reading compulsory for

all children. That narrow and restrictive application of the reading process has remained as a basic characteristic of our educational programs. Recognition that the reading process is utilized with all types of stimuli has been slow in developing. Consequently, it is particularly timely for this Council to express a concern for the reading which must be done with regard for the mathematical aspects of experience.

The Eighteenth Yearbook of the Council (36:2) presents a thesis that "mathematics should be taught as though the word were a transitive active verb." It also presents a correlative statement that "if students are going to 'mathematick', they will have to be given something to 'mathematick' about."

Both the thesis and the corollary are pertinent for this discussion. There is need to sense that reading is a "form of living" (26:5). Reading is a behavior process and not an area of subject matter (1). Fundamentally, reading is the process by means of which we sense our environment and make adaptive responses with regard for what is sensed. It is the process of making discriminative reactions with regard for any or all stimulation. That we will read is guaranteed by nature, but *what* we will read and *how well* the reading will be accomplished are matters largely determined by experience (10:15). In this discussion we are concerned with what one reads as he "mathematicks," and with how such reading can be accomplished more effectively.

Nature of Reading in Arithmetic

In arithmetic reading we are concerned with determining positions in time and in space, with the sizes and shapes of things, or with measuring how much or how many (20). The things we quantify, locate, or define as regards form or shape may be anything that we sense. We may read mathematically the odors, sounds, movements, light, heat, and any other stimulus with which we are concerned.

Arithmetic reading utilizes a procedure of *comparison*. Reading things and giving them meaning and significance as regards size, order, amount, shape, and position are primary characteristics of "mathematicking." Refining such reading and implementing it with standardized or conventionalized instruments of comparison

are important functions of mathematical instruction and learning. Determining:

How big is big, how small is small?,
How long is short, how tall is tall?,
How far is far, how close is near?,
What price is cheap, what cost is dear?,
How much is many, how few is some?,
When one may go, or when to come?,
What is the time, the month, the day?,
How numbers are used in work or play,
The tense of verb, the number of noun,
The shape of things: flat, square, or round?,
etcetera! etcetera!

is essential both for individual adjustment and for the stability and welfare of society. The development of instruments and procedures for making such comparisons constitutes a social contribution which is invaluable to man.

Mathematical reading occurs whenever quantity or relationships of a quantitative nature are experienced. Because these occur with great frequency, mathematical reading is characteristic of human behavior. In this connection it is pertinent to note that vernacular languages are structured on mathematical frames of reference. For example, sentences are required to show agreement between the subject and the predicate as regards *person* and *number*. Verbs are modified to express distinctions of time. Modifying expressions of various kinds are used to vary the basic meaning of the subject-predicate arrangement. In a very real sense all printed word reading as well as all spoken word reading is essentially mathematical reading. This was pointed out some 30 years ago by Thorndike (43), who wrote, "Understanding a paragraph is like solving a problem in mathematics." The fact that quantity and quantitative relationships are basic to language is also revealed by a study of the ideas which words symbolize. For example, Bobbitt (3:38) analyzed the idea-referents for the words listed in Thorndike's Word Book for Teachers; he commented as follows:

One of the surprising things is the way in which the *quantitative* element stands out in the vocabulary used. In part the terms were

numbers and other words referring to exact quantity. In still larger measure they were terms indicative of approximate quantity, such as "few," "several," "many," "much," "little," "large," "small," "abundant," and "meager." The evidence of language is that even in the pre-scientific age during which most of our language was evolved and most of the writings produced which were analyzed, the quantitative aspect of thought and action was an outstanding characteristic.

Enough has been said to indicate that mathematical reading is fundamental to communication by means of the vernacular. However, as refinements of mathematical discrimination occur, more exacting reading must be accomplished. To aid in quantitative reading, special languages are needed to serve with the more exacting communication.

Number Language a Special Language

Number languages are illustrative of the special mathematical languages. In order to read number language expressions one needs to know the pattern which governs the *spelling* of the symbols and the pattern which directs the symbol arrangement within the sentences. For example, decimal whole number symbols are spelled by controlled arrangements of the alphabet consisting of ten primary number letters, or digits. Our vernacular language similarly uses 26 primary letters. However, the vernacular language assigns more than one sound value to some of its letters and further complicates their reading by combining letters to represent sounds rather than words. The succinctness and simplicity of decimal symbol spelling and reading may well serve as a stimulus and as a guide for the rationalization of the spelling of the vernacular symbols.

Arithmetic reading entails the reading of both the number symbols and the vernacular symbols for quantitative amounts. The plan or scheme of decimal symbol formation is revealed by the symbol *10*. However the comparable scheme for the vernacular symbols is not clearly revealed before the symbol *sixty*. Since much of the reading of number values involves a transliteration from one of the languages to the other, the whimsicalness of the vernacular is a source of difficulty. Each decimal digit has a constant

value. That value is derived from its position within the primary scale of groupings of ones. The first digit, 0, symbolizes *not any*. The others represent the increasing increments of one. This primary value is expressed in the units' or ones' place in the numeral. The primary grouping represented by every decimal whole number is an aggregation of ones. However, a secondary grouping by tens is also incorporated in the decimal number pattern. Groupings of tens are indicated by placing the proper digit in a specified positional relationship as regards the ones' place. The symbol 10 illustrates both the primary and secondary aspects of the pattern for decimal symbol formation. It represents a grouping of ten and *not any* more. The digit 1 in the position given it represents *the group of ten*. The digit 0 is needed to show that not any more is intended. This may be more clearly recognized by analyzing the symbol 11. The 1 to the right is in the ones' place and, therefore, it represents its primary value only. The 1 to the left still retains its primary value but it also represents one group of ten. The whole symbol, of course, represents one group of ten and one more. That simple scheme applied over and over is sufficient to provide symbols for fantastic numerical expressions as well as for expressions which are used in our daily experiences.

Sources of Difficulty in Mathematical Reading

There are many sources of difficulty in mathematical reading. Consideration will be given to five of these areas.

First, the names of certain numerals are confusing. In comparison with the decimal system's simplicity, our vernacular pattern is confusing and difficult. For example, the symbol ten suggests no clue as to the nature of the next symbol, eleven. Twelve has no cue relationship to either eleven or thirteen. A pattern does show up with the teens. It is a pattern, however, which presents the aural cue in the reverse order for its written symbol. The three and five digits are given strange names. Beginning with sixteen the digits are given their usual names. Furthermore, that discrepancy recurs when the positional cues are disclosed with the third decade names. It is not until sixty that the digits are named by the names they originally were assigned.

Second, the fact that we use number languages patterned differently from the decimal system is a source of difficulty in mathe-

mathematical reading. For example, Roman notation uses a pattern which is substantially different from the one used in decimal notation. That is true also with regard to many of the tables of measures which we read. Mathematical reading might be greatly simplified if all of the measurements were made on systems based on decimal relationships. The nature of the number language symbols makes their reading substantially different from that which may be used in reading vernacular symbols. Each digit in the number symbol must be observed and read according to its positional value. This is particularly exacting when exponents and subscripts are used (45).

Third, the language forms used for expressing fractions and ratios are a source of difficulty in mathematical reading. The customary practice of referring to the elements of a fraction symbol as *numerator* and *denominator* gives little aid in interpreting such expressions. Fraction symbols are representative of quantitative amounts comparable to those symbolized by whole numbers. Fraction symbols differ from the whole number symbols by involving the idea of division.

Fourth, mathematical communication commonly involves the use of charts, maps, graphs, diagrams, and comparable devices for expressing location, size, order, or amount. Reading such expressions is a problem for concern in the teaching of mathematical reading.

Fifth, reading computational procedures is a specialized type of reading. Persons who know the bases for the steps in computation do such reading with a competency which is beyond that displayed by those who merely imitate or who compute by habit. Too little concern has been shown for developing intellectual reading of computation. *Tell-and-do* practices produce mathematical readers comparable to those who *bark at words* in vernacular reading, but who do not know what they read because they were not listening as they read.

Primary and Secondary Reading

Man is constantly confronted with situations which involve making discriminations with regard for positions in time or space, amounts in size or number, shapes or forms, and order within a sequence of things or events. These quantitative aspects give

mathematical reading a perspective it has not always had. Too often mathematical reading has been restricted to the special uses of number and other technical language forms. These are important, certainly, but there is need to recognize that they represent *secondary* levels of the reading behavior. The *primary* levels are concerned more directly with reading in the concrete. It is experience with the primary aspects of reading which gives the basis for ideas that make secondary reading feasible. The omnipresence of quantitative relationships in man's experiences is revealed by the ideas symbolized in his vernacular vocabularies, and by the many and varied devices which he has created for coping with quantitative measurements and communications regarding them. Viewed from this perspective one can assert with conviction that "arithmetic is a basic social study" (38).

Reading in arithmetic is "mathematically." Developing abilities to perform effectively in this regard must be the central function of arithmetic instruction and learning. It is unlikely to be achieved as a spread from other areas of learning.

"Mathematically" is most likely to occur when the situations confronted by the student pique his interest and challenge his intellect. Poor reading occurs when these conditions do not prevail. Hence, there is need to devise ways of making the quantitative aspects of human living more clearly recognized and appreciated. As the Eighteenth Yearbook of this Council so aptly said, if we want children to "mathematically," we must give them something to "mathematically" about.

ARITHMETIC POWER THROUGH READING

DAVID H. RUSSELL

Reading ability is a composite of power and speed. Many factors contribute to one or the other, and sometimes to both. For example, accurate perception may increase speed, and related background of experience may add to power in reading. Precise perception may also add power in word recognition, and familiarity of concepts may contribute to speed in perceiving symbols. Both speed and power in reading, accordingly, are dependent upon a group of complex and often interrelated factors.

So with arithmetic. Speed may depend upon accurate perception and rapid computation, power upon understanding plus ability to do relational thinking. But ability to see relationships may increase speed, and accurate perception and computation may add to power. A complex pattern of Speed-Power-Achievement is true of both reading and arithmetic.

De Quincey once wrote, "All that is literature seeks to communicate power; all that is not literature, to communicate knowledge." Without detracting from the values of literature, the present chapter assumes a more functional definition of power than enrichment of personality through literature. It assumes that knowledge is power but goes along with T. H. Huxley that "The great end of life is not knowledge but action," at least in the fields of reading and arithmetic.

Power in arithmetic relates to mastery of skills, understanding of quantitative concepts, and ability to go through the various stages in the solution of problems of increasing difficulty. The intricate pattern of Knowledge-Skills-Power-Action runs through much of the arithmetic curriculum.

If power in arithmetic is closely related to power in reading, some idea of this latter term is needed by the teacher. Power is close to comprehension, broadly conceived. It relates to the ability to grasp meaning, to understand—but it is more than literal comprehension. Durrell (15) has stated that comprehension is the ability to translate printed symbols into images, ideas, emotions, plans, and action. It may also include the ability to evaluate for accuracy, to check critically for bias, and to organize given facts in terms of purpose. The use of symbols, the ability to examine critically, and the need to organize in terms of given data, requirements, or purposes would all seem to be important not only in reading, but in arithmetical activities.

Power Patterns

Children illustrate power patterns in arithmetic in many specific ways. Alice can check the group's lunch money. John explains to the class in current events the newspaper story that flying over the North Pole saves time for travelers to Europe. Louise can bake double or triple the recipe when she is bringing a cake for a school party. Tom can figure the difference between the legal

limit and the size of the present national debt. More prosaically, Betty solves a verbal problem in the arithmetic text with assurance and zest and understands the succeeding illustration of a variant type of problem. In short, arithmetic power has many facets and components.

Attempts to describe children's thinking in arithmetic are still largely on the theoretical level but a schema may be attempted, for example, in a verbal problem situation. Suppose the pupil is faced with the following problem:

Bill collects \$40 weekly on his paper route but must give \$34 of this to the circulation manager of the daily paper. If Bill saves all of the money that remains, how long must he deliver papers so he can buy a second-hand bicycle for \$72?

B. has saved money; has practiced arithmetical computation, etc.	B. has friends on paper routes and hopes to have one himself next year.	Reads problem and grasps idea of saving some money each week.	Rereading and thinking. Insight re two main steps as <i>capture of meaning</i> .	Solving by subtracting, then dividing.	Checking validity of solution.
Background of experience	Knowledge of concrete situation	First survey reading	Rereading with insight and planning	Steps in actual solution; reading one's own writing	Rereading for evaluation of process and result

This outline suggests the intimate relationship of reading to the process of problem solving, especially if reading is regarded as involving background meanings, comprehension, interpretation, and organization of verbal materials.

Previous Writing

The central role of reading in much school learning has been documented in research and in general articles. For example, Horn's classic chapter (21) and Michaelis' more recent discussion (30) testify to the importance of reading in the social studies. Similarly, previous discussions of the role of reading in certain

arithmetical situations have been written by Brueckner and Grossnickle (6:498-501), by Morton (32), by Rosenquist (33), by Spencer (39), and by Wilson (47). The importance of meaning in both reading and arithmetic has been underlined in an article by Buckingham (7) and a recent volume edited by Stauffer (41).

In addition to the books on arithmetic that discuss the role of reading ability, certain books on reading instruction give suggestions for the improvement of reading in mathematical situations. These include such texts as those of Bond and Wagner (4), McKee (28), McKim (29), and Strang (40). These sections typically discuss reading skills needed in arithmetical problem solving and some offer specific suggestions for practice.

Research on Reading in Arithmetic

Research on the role of reading abilities in arithmetic may be classified into three somewhat overlapping areas: (a) arithmetic and general reading ability, (b) arithmetic and vocabulary, (c) arithmetic and specific reading skills. The following summary indicates that the second and third areas have been most fruitful of positive results, but that research findings can be considered fairly adequate in only the second of the three divisions.

In the first area, studies dating back to the 1920's indicate positive correlations between reading ability and problem-solving scores. These correlations usually occur in the .30 to .60 range, depending upon the tests of arithmetic and of reading used. As early as 1918 Monroe (31) showed that the same problem, involving identical quantitative relationships, could be stated verbally 28 different ways in arithmetic texts and other sources. Obviously reading was involved in the different statements of the problem. The fact that the correlations, however, are not very high led Stevens (42) in 1932 to the conclusion that "ability in the fundamental operations is more closely correlated with ability in problem solving than is general reading ability." In grades 4 to 8, Coffing (11) found a median r of .50 and no pattern of relationships between silent reading and ability to solve verbal problems from grade to grade. In grades 5 to 8, Morton (32) found problem-solving ability correlated with intelligence .78, with skill in fundamental operations .70, but with general reading speed only .23.

A number of studies quoted below show higher correlations between arithmetic scores and specific reading abilities than between arithmetic and general reading ability. Because general reading scores often include tests of paragraph reading of literary materials for main idea and for unrelated details, the lack of a striking relationship is not unexpected. General practice in reading, accordingly, may be criticized as an ineffective form of arithmetic teaching.

In contrast, studies in the second area of the relationships of arithmetic and vocabulary, especially mathematical vocabulary, indicate a consistently positive and strong connection between the two. Cole's lists (12) are a standard reference on basic vocabularies but need revision after 20 years of social change. Concepts of cardinal and ordinal numbers, of mathematical processes, of the number system, of weights and measures, of a thousand uses of arithmetic as in *perimeter*, *installment buying*, *profit*, and *group insurance* may pop into the elementary school child's arithmetical activities. Knowledge of these specific concepts seems consistently related to problem-solving abilities. For example, Foran (18) found that technical terms and other unfamiliar words at a given age or grade level interfere greatly with problem-solving performance at that level. Eagle (16) found that mathematics vocabulary is closely related to mathematical achievement in the junior high school and J. T. Johnson (25) believed that vocabulary is the main factor in problem solving, ranking ahead of reasoning ability. In a study involving 598 seventh graders H. C. Johnson (24) gave practice on selected arithmetic vocabulary for 14 weeks. The practice included word drills, use of dictionary, and keeping individual notebooks for difficult words. Mimeographed material supplemented the text. He found that the experimental group given vocabulary practice made significantly higher gains not only in arithmetic vocabulary but in problem solving which involved some of the words taught. He concluded that training in vocabulary should be made an integral part of arithmetic activities from the time the child begins formal work.

The need for direct attack on selected skills and abilities is illustrated also in the third area of research dealing with relationships between arithmetic and specific reading skills. As early as

1925 Lessenger (27) found that reading instruction had favorable effects on arithmetic computation scores. Analysis of initial and final scores showed that before reading practice, the average pupil in grades 3 to 8 was penalized approximately six months because of faulty reading. After reading practice over eight months this loss was reduced to about half a month with consequent improvement in reading scores. Among other conclusions Lessenger stated that "the desirability of specific training in the reading of signs which show the operation... is clearly indicated."

In a later study Hansen (19) found significant differences between good and poor problem solvers in general language ability, in vocabulary and in specific reading skills. Perhaps the clearest evidence favoring specific work in reading is that of Treacy (46). As stated above, most of the reading tests used in the study were of a verbal type with little quantitative content. In a sample of 244 seventh-grade pupils Treacy found that good achievers were significantly superior to poor achievers in reading skills and mental age. With the effects of mental age removed, he found no significant differences on the following six abilities: predicting outcomes, following precise directions, rate of comprehension, general information, getting central thought, and interpretation of content. However, he found significant differences between good and poor problem solvers in nine reading and language abilities. Those significant at the 1 percent level were quantitative relationships, perception of relations, vocabulary in content, and integration of dispersed ideas. Those significant at the 5 percent level were arithmetic vocabulary, isolated words, clearly-stated details, drawing inferences and general reading level.

The particular reading and language tests which Treacy found related to problem solving are of distinct importance, especially in the light of a study by Fay (17) of the relationships between specific reading skills and arithmetic achievement. (He also studied achievement in social studies and science.) Fay found, when the effects of chronological age and mental age were eliminated, that arithmetic achievement was not specifically related to a group of reading abilities. The apparent conflict between this and such studies as Treacy's is partly explained, however, when Fay's reading tests are examined. He included tests of (a)

reading to predict outcomes of given events, (b) reading to understand precise directions, (c) general reading comprehension, (d) reading of maps, graphs and charts, and (e) use of the index, references, and the dictionary. The important point here is that these are not the same reading abilities as those tested by Treacy. It seems safe to conclude, then, that problem-solving ability is not related to *any* specific reading skills. Rather, it is closely related to certain reading and thinking abilities such as general and mathematical vocabularies, ability to grasp quantitative relationships, and ability in drawing inferences and otherwise integrating scattered ideas. Arithmetic is not related to reading to predict and to nonspecialized ability in using reference materials. Furthermore its relationship to general level of reading comprehension and to ability to read for details seems contradictory in different studies. In general, reading abilities closely related to quantitative thinking are the ones that count. The specific abilities needed appear more clearly in the analysis of problem solving in the next section.

Specific reading skills may also include the reading of numbers and other mathematical symbols and the use of reference books to locate and organize information about the origins and uses of mathematics. As Spencer points out in the first part of this chapter, reading ability may be concerned with a wide variety of symbols and situations. The currently active study of the psychology of perception should give us new hints about the reading of mathematical symbols. Research on reading numbers, such as that by the Wheelers (48) mentioned below, is in short supply. Since reading numbers and mathematical symbols is not word-calling but a power process involving understanding, the discussions in other chapters of developing quantitative concepts apply directly to arithmetical reading. For example, the modern child encounters large numbers in a news story on the cost of building freeways or the rise of the national debt. In such cases an understanding of our number system is essential to reading of symbols.

The understanding of mathematics and its role in modern life may be increased through the use of reference books and trade books dealing with scientific and mathematical topics. A feature



of the publishing of "juveniles" in recent years has been the large number of such books in series and as individual titles. These attractive books include such series as the *Real Books* (Garden City), the *All-About Books* (Random House), the *Gateway Books* (Random House), and the *First Books* (Watts). Standard children's encyclopedias and a new 17-volume collection *The Wonderful World of Science* (Spencer) contain many articles which include mathematical concepts in meaningful settings. Individual books on television, chemical experiments and other such topics ordinarily give practice in reading and understanding mathematical concepts. Trade books ranging from *How Big is Big?* (Scott) to *Fun With Mathematics* (World), *The Wonderful World of Mathematics* (Garden City), and many mathematical puzzle books give specific materials that challenge the superior reader and the budding mathematician. With such materials the teacher's methods of enriching the reading program will enhance understanding of arithmetical concepts.

The Psychology of Problem Solving

It seems likely that the solving of verbal problems will continue to be an important part of school arithmetic. Accordingly, a quick look at the psychology of problem solving will give additional leads, not only to the process itself, but to the reading skills which are most closely allied with it.

Useful summaries of research about problem solving have been presented by Brownell (5), Johnson (22, 23), Russell (34) and Thorndike (44). In addition, many individual studies such as those of Bloom and Broder (2), Burch (8), Buswell (9), Doty (13), and Duncker (14) give psychological insights into the process. In different terms, but with considerable agreement, these research workers find in problem solving a group of skilled and interrelated activities marked by relational thinking in a variety of patterns. These patterns may involve trial and error (especially in puzzle situations), sudden insight, or gradual reorganization, different behaviors being evoked by different types of problems at different levels of difficulty.

The different writers who discuss problem solving make it a systematic and logical process containing from three to nine steps. Eight such analyses are summarized by Russell (34:256). Detailed studies suggest, however, that even skilled problem solvers seldom follow a regular progression of steps, such as outlined by Dewey, Humphrey, or Polya, but that they engage in such activities as *getting to understand the problem*, *search*, *suggesting solutions* and *eliminating sources of error* in almost any order. The research suggests that problem solving is not a unitary factor, best described by one term such as *reasoning*, but rather a complex of different abilities. While the specifics are not always clear, the essential parts of problem solving seem to be an orienting function, an elaborative and analytical function, and a critical function. The problem-solving process varies with the nature of the form of stating the problem, the methods of attack known by the solver, the personal characteristics of the solver, and the total situation in which the problem is presented.

This summary of the process suggests that reading abilities may play an important part in solving verbal problems. The

studies mentioned suggest the hypothesis that reading is involved in the following items:

1. The nature of the material read
 - (a) Symbols
 - (b) Concepts stated or implied
 - (c) Form of stating the problem
 - (d) Addition of irrelevant material
 - (e) Lack of information needed for solution
2. Characteristics of the individual solver
 - (a) General attitudes toward arithmetic
 - (b) Motivation and specific set
 - (c) Background knowledge of setting and processes involved
 - (d) Mental ability in terms of seeing relationships or perceiving patterns
 - (e) Computational skills
 - (f) Critical attitudes toward answers
3. Reading-thinking abilities
 - (a) Ability to apply past experiences to partially new situations
 - (b) Ability to read for several specific purposes such as noting details, grasping sequence, and determining the chief problem
 - (c) Ability to restructure—to analyze into facts given, processes required and outcomes expected
 - (d) Ability to differentiate into a series of subordinate problems with some final integration of subordinate solutions
 - (e) Critical evaluation of solution in relation to experience and to details in the problem.

This comprehensive list is not likely to be of direct help to Miss Brown working with 30 children in the fourth grade for 190 days of a school year. Each item may involve or affect reading. Each item in the list needs to be broken down into a series of activities which allows for gradual growth of understanding plus practice plus recombination with the total process of problem solving. Fortunately, a few studies give more specific suggestions. For example, Doty (13) found that fourth and sixth graders used

procedures resulting in incorrect solutions such as (a) determining methods from the numbers in the problem and from verbal clues, (b) neglecting the questions asked, (c) disregarding the data given, (d) selecting a process appropriate to only one part of the problem, (e) working one step before reading ahead, and (f) estimating the size of the answer and then juggling the figures to get that answer. Such negative examples suggest positive techniques to be taught.

Fortunately, Doty was able to suggest four factors which involve reading and which are related to success in problem solving. The attainment of correct solutions was associated with (a) rereading the problem before computing it to check one's comprehension of it, (b) checking correctness of an answer by a final skimming of the whole problem, (c) evaluating reasonableness of answers, (d) visualizing the problem, and (e) recalling similar problems. Such statements indicate subtleties in the reading of problems which the busy teacher of 30 youngsters cannot always be expected to utilize. However, the list of reading skills and situations above and the Doty suggestions offer much help to the teacher working with a few children and to supervisors and other curriculum workers who can plan inservice workshop activities, ways of utilizing teachers' manuals, and curricular materials for bulletins and courses of study. A few samples of such suggestions are given in the next section.

Reading Activities and Devices

Reading involves quantitative symbols, special arrangements of symbols and words as in tables and verbal problems. This report has concerned itself chiefly with verbal problems but no teacher or child can neglect the non-verbal phrases described by Spencer. For example, the Wheelers (48) have indicated difficulties of first- and second-graders in learning to read numerals. They believe that games and devices may have considerable value in developing accurate perceptions.

Between symbols and verbal problems lies the intermediate group of reading materials in tables, charts, and graphs. Many arithmetic texts and their accompanying teachers' manuals give specific suggestions for teaching such materials. Most children

do not develop independently in their abilities to interpret tabular information or circle graphs, and so specific and perhaps individual help is often needed.

Suggestions for reading verbal problems have been made by Smith (37), Schubert (35) and others. A few of these suggestions may be combined into a list like the following:

1. Help children to use the first reading to visualize the problem-situation as a whole.
2. Direct attention to the question usually stated near the end of the whole problem.
3. Have pupils reread the problem to analyze it into a series of steps necessary for a solution.
4. Give practice in estimating reasonable answers after a second or third reading.
5. Have pupils select the first process and write their first statement only after two or more readings of the problem.
6. Give help in spotting irrelevant sentences.
7. Give specific help in building understanding of quantitative terms like *numerator* and *acre* and of processes such as obtaining a batting average or calculating a percentage profit. Vocabulary games may help.
8. Without giving numbers have children state how they would solve problems such as, "Tom is a boy who got three separate birthday gifts of money. He wants to know if he has enough money to buy a basketball. What should he do?"
9. Have children rewrite problems in simpler terms to try out on other members of their group.
10. Have pupils list some of the purposes for which they read (main idea, details, directions, fun, etc.) and decide which are most useful in solving verbal problems.
11. Encourage the reading of stories involving quantitative situations such as *Five Puppies for Sale*, *The Five Chinese Brothers* and *Millions of Cats* in the primary grades. Such books as Bendick's *How Much and How Many* may challenge fifth and sixth graders, and science fiction may appeal to the seventh to ninth grade levels.
12. Have children bring stories clipped from magazines and newspapers which involve mathematical data. These may involve

the flying log of a prominent world traveler or the calculation of the speed and orbit of an artificial earth satellite. Here the procedure is understanding plus criticism plus use where possible.

This article has suggested that power in reading arithmetic problems may be impaired by inability to comprehend in whole or in part, specific recognition errors of words and phrases, and confusion in thinking, especially in attempts to organize the material read. The 12 suggestions for possible ways of overcoming such difficulties have not all been confirmed in research studies and are simply examples of what the resourceful teacher may try with different individuals or groups to improve their reading of mathematical materials. With the more abstract suggestions of the research reported above, they constitute a program for improving, through reading, children's power in arithmetic.

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Instructional Materials

CLARENCE ETHEL HARDGROVE and BEN A. SUELTZ

A GOOD INSTRUCTIONAL program in arithmetic provides a sequence of stimulating experiences of an increasingly mature nature from which a child can learn. The teacher should choose instructional materials that will help make the experiences vital. It is the purpose of this chapter to consider the importance of instructional materials to the experiences of learning, to suggest uses and limitations of these materials, to consider principles of selection, and to suggest specific materials for use at different grade levels.

BRIEF NOTE ON LEARNING

A child grasps the ideas of arithmetic by putting his many experiences into a meaningful context and by reorganizing selected elements to formulate generalizations. According to McConnell (9:282) learners should "have a wealth of experience . . . in bringing understandings, information, abilities, and skills from many sources . . ." The experiences and the material of the

experiences do not furnish ideas as an end result; it is only as these are related to purpose and reflected on by a child that learning results.

The importance of the role of instructional material to learning is emphasized by Van Engen and Dale. In discussing the formation of concepts, Van Engen (12:86) states that

reactions to the world of concrete objects are the foundation stones from which the structure of abstract ideas arises. These reactions are refined, reorganized, and integrated so that they become even more useful and even more powerful than the original response.

Dale (3:34) affirms this as he lists the two elements involved in concept building: "(1) Concrete experiences and (2) the ability to combine and recombine these experiences in many ways."

The process of experiencing which results in the formulation of ideas has been shown to consist of four levels of abstraction (5:156, 1:402-408, 6:169). They are periods of *readiness*, *exploration*, *verbalization* and *systematic generalization*.

Readiness is described by Grossnickle and his associates (5:157) as

that period in the learning situation when the child's background for learning a new concept is appraised, the foundational experiences are provided, and a purpose for the new learning is established in the mind of the learner.

This period is not definite as to time and place; it may take place outside the classroom and without the guidance of a teacher. It is necessary, however, whether unplanned or planned, whether unguided or guided. It is important for purposes of making the child familiar with the elements of the experience and interested and involved in the situation.

The *exploratory period* follows readiness experiences and is characterized by guided experimentation with sensory material to help a child develop arithmetical ideas. It is a period of doing and observing. Russell (10:131) says that children

are dependent upon many opportunities to manipulate and explore—an environment rich in blocks of different sizes and colors, toys that go, equipment that can be pulled, pushed, reconstructed and climbed.

Such experiences seem to be a necessary background for any verbalization of mathematical concepts

Experiences and materials do not in themselves result in learning. They must be purposeful and must result in some kind of organization.

A *period of verbalization* represents a time for recording and discussing observations of a child's response to his exploratory period. The observation of experimentation with the materials of the exploratory period takes the form of verbalizations which describe in oral or written word, or in nonverbal symbols, the results of the exploration. It is a period when the materials of a situation may be put aside for symbols. These symbols may take the form of sounds, charts, pictures, written words, and other written symbols, or combinations of these.

The *systematic generalization* results from periods of exploration and the verbalization of observations. The culmination of the learning experience is the formulation of a generalization. This is done by consideration of all related experiences, elimination of unrelated elements, abstraction of the likenesses or differences, and the organization of related details. In the process, Russell (10:15) says, "certain relationships are perceived and a number of details, perhaps unrelated previously, resolve themselves into a possible solution or conclusion." The conclusion, an arithmetical idea, then becomes a part of a child's concept of arithmetic.

MATERIALS OF INSTRUCTION

For each period of learning there are instructional materials that aid in the process as shown below.

Levels of Abstraction in Learning	Material of Learning
1. Readiness	1. Exploratory (2:97)
2. Exploration	
3. Verbalization	2. Pictorial
4. Systematic	3. Symbolic
Generalization	

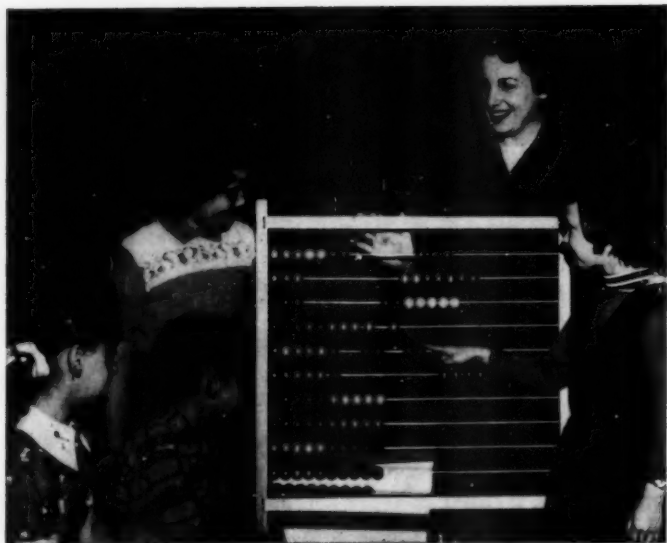
Exploratory Material

Exploratory materials are those materials of instruction used by children during periods of readiness and exploration. They are not only visible but most of them may be felt, touched, and handled. They are the non-reference, non-pictorial type. This classification contains both real objects that may represent an idea and objects that are useful in daily living.

Real objects that may be used to represent an idea are of many types. Paper circles cut into fractional parts are used as exploratory material to find that $\frac{1}{2}$ is larger than $\frac{1}{4}$, that $\frac{1}{2}$ and $\frac{2}{4}$ are equivalent, and that $1\frac{1}{2} + 1\frac{3}{4} = 3\frac{1}{4}$; a number scale is used to find that three 4's are 12, and that 5 and 8 are 13; and, an abacus is used to show the regrouping process in the addition of 25 and 37 or the subtraction of 15 from 52.

Exploratory materials that are useful in daily living are of two types. The first type includes all the objects in the pupil's world that may be used to count, compare, and describe. Examples of these are the chairs in the classroom, the boys and girls attending the school, the lot on which the school building is located, the crayons in a box, and the buttons on a shirt. The second type are the instruments of measure which boys and girls need to learn to use intelligently. It is important that as many as possible of the simpler instruments be in the classroom for use by the children. Clocks, scales for measuring weight, measuring cups, thermometers, and yardsticks are examples of these.

Exploratory materials have an especial appeal because they enable a pupil to combine seeing, handling, and talking about the factors involved in a situation. This encourages thinking and discovery and reduces the tendency to memorize a generalization. A child who sees a collection of five pebbles and another of four and then places them together into a new collection of nine has not only established the conclusion, but he has constructed a situation to enable him to discover the real meaning of addition. Similarly, one who has worked with fractional cut-outs and has developed the ideas and language of fractions before he has learned to work with symbols for fractions has a better understanding of the role of fractions than he could have by working solely with symbolic materials. Young children particularly seem



to learn via the kinesthetic sense, and they enjoy the actual handling of materials.

Selection of Exploratory Material. Materials of this classification may be purchased from a commercial jobber or they may be teacher- or pupil-made. The source will depend on the cost, the durability needed, the simplicity needed, and the educational benefits to be gained by children from the construction of the material. The durability needed for an abacus used for classroom demonstration may suggest the purchase of a commercial product. The excessive cost of a balance scale and the simplicity of construction needed suggests a teacher-made instrument. The educational benefits that children get from preparing their own fractional cut-outs would rule out the purchasing of these.

A teacher should be alert to purchase only materials that are useful and helpful in enabling a child to learn. Many commercial jobbers advertise their products without regard to the use to be made of the material. The classroom should not be filled with interesting appearing gadgets which have little educational value from the viewpoint of producing growth in arithmetical ideas.

Material should be readily available for effective use and selected in terms of its storability. A demonstration abacus should not be so large or so constructed that it cannot be quickly moved aside for activities that do not require its use. A number scale should be made flexible so that it can be folded and put away when not needed. However, all materials should be so stored that the teachers and pupils may find them immediately when they are needed.

The Use of Exploratory Material. Exploratory materials have three uses. First, and very important, is their use in the development of mathematical ideas; second, in the verification, reinforcement or expansion of an idea; and third, in other problem-solving situations.

The first use of exploratory materials, the development of an idea, is illustrated by a six-year-old who counted four chairs and two chairs, four tongue blades and two tongue blades, and four jackstones and two jackstones, before making the generalization that four and two are six. It is also illustrated by a fourth-grade boy who separated 13 cubes into groups of 4 cubes to find how many 4's in 13, by a fifth grader who found that $\frac{3}{4}$ and $\frac{3}{4}$ are $1\frac{1}{2}$ by pouring water from a measuring cup into a pint measuring cup, by a 13-year-old who compared gram weights to a pound weight to find a relationship. In these examples, the boys and girls were obtaining information from experiments with exploratory materials and as a result were helped to make generalizations.

Exploratory material has as its second use that of the verification or the reinforcement of an idea. A mathematical generalization is questioned by a pupil; then he uses material to prove or disprove the conclusion. A 7-year-old may need to use counters occasionally for assurance that 5 and 4 are 9. A 9-year-old may need to prove that $1\frac{1}{2}$ and $\frac{3}{4}$ are $2\frac{1}{4}$ by use of fractional cut-outs. A 13-year-old may question the application of the generalization, "The circumference of a circle is π times the diameter," to circles that are larger than those he measured in the classroom. He experiments with materials to verify the generalization.

The third use of exploratory materials is in problem-solving situations other than those previously described. This is especially true when the mathematical ideas needed have not been gener-

alized. In such an instance, a reason for observation and generalization is provided. A primary child who needs to know the cost of two 7¢ stamps for mailing letters to friends may use milk bottle tops for cents. He can make two groups of 7 counters and count to find the result. Fourth-grade children find the area of the classroom floor by counting the number of square feet of newsprint used to cover the floor. An eighth-grade boy uses a protractor to find the size of an angle for use in a problem situation.

It is important for a teacher to know when to use a specific material to good advantage, but it is equally important that he know how long to use it and when it has ceased to be important for an individual and for a group. It is as wrong to continue the use of a counting frame for finding addition combinations after a child has made the initial discovery and understands it as it is to assign examples in subtraction daily when the child is very adept at the process.

Whatever the use, the material should be used only temporarily. Its use in the development of an idea, the verification of an idea, or in other problem-solving situations should be brief, being put aside as the child develops a more mature way of thinking.

Pictorial Material

Pictorial material includes visual material in the form of pictures and photographs, diagrams, charts, graphs, tables of values, filmstrips, films, and kinescopes. This type of material includes anything of a pictorial nature that will help a child in the development of an arithmetical idea. Pictorial materials are used during the verbalization level of learning and to some degree in the exploratory period.

Books and magazines are the source of many materials of a pictorial nature. These materials range from simple pictures, to show a group of five, for example, to the illustration of a complex mathematical formula. Some teachers keep a classified file of pictures and illustrations that may be used directly for instruction or placed on a bulletin board to raise a problem or create interest. Frequently teachers ask pupils to bring to class pictures that illustrate a mathematical idea such as addition, percentage, or a right triangle.

Graphs and charts are readily available from many sources. All of them will tell a story which may be investigated by a class. The investigation may be as simple as interpreting the data given, or it may involve checking the sources of data and speculating about inferences to be drawn. With information from graphs collected by the children a teacher may find it worthwhile to combine the activities of reading, composition, and social studies.

Of special note in recent years is the wealth of projection material that has been placed on sound films, filmstrips, and kinescopes. A kinescope or movie is particularly valuable to show a change in some factor of a situation, such as the increase in earnings of a business over a period of years. Films are also very useful in showing the operation of a business which cannot readily be visited at first hand as, for example, a clearing house. A good film can present materials which a teacher may not have readily available and which an expert can demonstrate with greater effectiveness than the teacher. However, it is a mistake to rely solely on films when the same ideas can be learned through use by the teacher and children of the same materials as those used in the film.

The Selection and Use of Pictorial Materials. In the classroom, "a picture equals a thousand words" if it is a picture that tells a story and is used at the opportune time and in such a way that its purpose is clear. For example, a picture showing five boys with five bicycles and five dogs may be used in kindergarten or grade one to develop the *five-ness* idea. In the intermediate grades, pictures showing fractional parts of a cupful, glassful, yardstick, and pie will do much to help pupils understand that fractional parts of real things need not be restricted to the circles commonly used in teaching.

Pictures are most useful when they depict something that cannot be brought into the classroom or when they are used as a device to create interest and speculation about an object or situation. A picture of a hen and chicks is more valuable than the real chickens in an arithmetic classroom. Several real objects of different size and shape that weigh one pound each are better for the development of the concept *pound* than a picture of these. How-

ever, a picture of various objects that weigh a pound can lead to speculation about weights and then to a need for the real articles.

Pictures may be used for bulletin board displays to show mathematics at work in a business, trade, or profession as, for example, pictures to show a carpenter's use of measurement. A great number of geometric ideas may be illustrated by pictures, diagrams, and cartoons. These may range from illustration of simple straight lines and circles to illustration of the equations of the orbits of satellites. One never knows what materials may provide an interest for a particular child; therefore, a wide range of pictorial materials is suggested.

Charts and graphs are used much as other pictorial materials. Many of these graphic materials are available in magazines, trade journals, and manuals. In any classroom the parents of the pupils have a considerable range of employment and interests and can help to provide materials that will make arithmetic meaningful and interesting.

Projection material, which includes filmstrips and motion-picture films, is available on many topics of arithmetic. The topics range from the simple development of the idea of numbers, such as the meaning of four, to more advanced development of percentage and elementary algebra. Some of the projection materials do pictorially what each teacher should do with exploratory materials. These materials are most useful, therefore, for a review-summary of a topic, as, for example, one that illustrates the *how* and *why* of division.

Films and filmstrips which describe institutions and historical developments are very effective in a classroom because the maker of a film has available and can utilize materials that cannot be brought into the classroom. Such a film, by its superior dramatic presentation, can be invaluable in stimulating interest and in providing sidelights of information. The story of weights and measures, with historical development, can be made much more real and alive on film than by common classroom procedures, as can the operations of a stock exchange. Superior organization, coupled with an excellent presentation that has dramatic appeal, makes a good film or kinescope a valuable teaching instrument. But it is not enough to show the film; the values come when the

children, under the guidance of the teacher, consider the information given. This consideration, stimulated by the film, should result in the raising of the level of the children's insight.

The use of television as a teaching instrument is still in its infancy, but the possibilities appear to be worthy of study. An expert teacher who has carefully planned the development of an idea with attention to the age-level of the audience can do a superior job of teaching each day, but a teacher is still required in each classroom to give additional instruction and guidance to the individuals of the group. In a non-statistical, exploration-demonstration of the teaching of arithmetic by television in Pittsburgh (11), it was noted that the classroom teacher could not be replaced by a teacher appearing on television; that the planning and teaching of a superior teacher resulted in a vital lesson each day; that the ability of cameras to focus on points of interest contributed to learning situations; that programs of this type had possibilities for creating public understanding of teaching methods; and that the programs could be a tremendous force in in-service teacher education.

In a few years we should expect to have many good TV kinescopes available for use in arithmetic classrooms. As a warning, these kinescopes, as other projection materials, should be used only where they are genuinely helpful. They should not be expected to replace the more intimate relationship of teacher and pupil and to produce the very great values that come from a teacher who is a stimulating guide-counselor-director-evaluator of learning.

Symbolic Material

Symbolic material includes all printed and written matter which gives information. This material is in the form of systematic generalizations. It is growth in the use of symbolic material at which all teaching is aimed. Reference materials and textbooks are the major forms of symbolic materials.

Reference Materials. Every school should have available for each classroom a number of reference materials which range from simple number stories to encyclopedias. Included should be many

of the delightful books (7) (8) now being published for children that help to develop mathematical ideas, newspapers and magazines, mail-order catalogs, reference books such as *The World Almanac* and *Britannica Junior*, government bulletins, bulletins published by insurance companies and banks, and books written about the history of mathematics.

The wise teacher will know a great deal about the backgrounds and interests of his pupils and will place them in contact with reference material to help them develop a particular interest in mathematics. A man who is now program director for an electronic computer determined his career when, as a junior-high-school student, he asked his wise teacher if a machine about which he was reading used the same method of division as he did. By use of reference material and with the guidance of his teacher, the boy mastered arithmetic in the system of numeration in base two. The reference material was the source of a reason for his learning and of information for the development of ideas.

Frequently a hobby or an out-of-school activity such as a 4-H club, can be the touchstone which stimulates a boy who has been called a *slow learner* to develop an interest and a will to achieve in arithmetic. Harold had been assigned to an opportunity room because he just didn't seem to *get* arithmetic. Two years later, because of a teacher's guidance, the boy received a mark of 92 in a state examination. The teacher had capitalized on the boy's interest in cattle as a reason for learning, had provided him with much reference material, and had challenged him to find ways of solving quantitative problems for which he needed answers.

Many books (7) (8) of an informational nature have been written describing social institutions which require an extensive use of mathematics. These are of special use in the classroom and are written about such topics as money, banking, the calendar, and the development of standard weights and measures.

The wise teacher will have available, in a classroom, reference material that will help enrich the arithmetic idea being developed, that will help a child find an interest in arithmetic, and that will help him expand an interest he already has.

Textbooks. The arithmetic textbook is the most commonly used instructional material in our schools. The textbook should serve

as a combination learning and reference book. The great value of learning from a book should not be overlooked because this form of learning is a necessity for adults, who frequently must consult a guide in order to understand a new subject such as cooking, masonry, or cabinet making. The text is also invaluable for pupils to use as a reference book in refreshing an idea or in finding how to do something that has not been taught. Workbooks usually serve to provide extra practice materials, but some are really a combination text and workbook and attempt to serve both purposes. The modern teachers' manuals which accompany most textbooks series are an invaluable source of ideas for beginning teachers and also for the more experienced ones who should be expanding their concepts of teaching.

The great weakness of a textbook, and of any printed matter, as instructional material is that it must rely upon printing which often presents a systematic generalization before a child has had an opportunity to explore and verbalize his observation. It also is restricted to words, pictures, and diagrams in the development of ideas. It cannot contain real fractional parts of apples, real coins, or other exploratory material. The user of the textbook must depend on pictorial material presented there and mental images of real or vicarious experiences. The wise teacher will supplement the use of the textbook with real and exploratory materials and guide the child to effective use of the pictorial and symbolic material of the textbook.

The Selection of Textbooks. It is agreed that most standard arithmetic textbooks used in the United States are carefully prepared in content, organization, and physical structure. The prime consideration is whether or not a particular book or series is useful to the teacher in helping pupils learn the arithmetic they should learn. To make a wise selection of a textbook requires discernment and perception and careful study of the book in relation to the purpose of teaching arithmetic held by the school. Usually a committee choice is better than that of an individual, especially if the book is to be used throughout a school or system. Textbook rating scales have not proven satisfactory when a total score is used because one factor that might make a book unacceptable will not have a significant effect upon the total score. In the

consideration of a textbook series it is desirable to trace the development and treatment of a specific topic through several books of the series.

A textbook series should be examined along with the teachers' manuals. Often the manual enables the reader to get insight into the plan of the books. If a school has many beginning teachers and a number who are returning to the profession after a lapse of several years, it would be very wise to choose a series that has good teachers' manuals. These manuals are in essence *advice books* for teachers who should be refreshing and expanding their understanding of arithmetic.

More specifically, it is worthwhile to check the textbooks and the teachers' manuals in terms of items such as the following:

1. Development of ideas, meanings, and understandings
2. Presentation of major steps in a sequence
3. The nature of graded exercises and problems
4. Provision for individual differences (*depth* and *extension* for fast learners and success experiences for slow learners)
5. Appropriateness of illustrative experiences used for development of ideas and of problem material for the grade level
6. Appropriateness of language and sentence structure for the grade level
7. The author's beliefs about learning and the teaching of arithmetic
8. Usefulness of the text to children as a reference book
9. Usefulness of the text and the manual to the teacher in evaluation of children's progress
10. Provision for the use of supplementary learning experiences by the children
11. The physical features of the text and manual (print, page format, diagrams and illustrations, and durability).

The way a teacher uses a book will be a factor in the choice. Some books surpass others in practice materials, and still others have a superior method of developing ideas. One book might be more useful as a pupil self-help book and another as a guide to a beginning teacher.

The copyright date and the advertised claims of a book com-



pany are not necessarily criteria as to the up-to-dateness of a textbook. Similarly, tests of readability may be misleading. Only a careful and critical examination of the contents will show how well a book may be expected to do the job a particular school believes should be done.

The Use of a Textbook. Textbook-type materials may be used as soon as a pupil is able to read and comprehend the ideas expressed by non-verbal symbols and/or words. For most pupils this comes in the latter part of the first grade or early in the second grade. However, there are many arithmetic ideas that should be developed prior to the reading stage and should be continued at all levels after children read and use textbooks. The whole realm of counting, the concept of addition as combining, of subtraction as removal or separation, and simple multiple grouping as multiplication should be done with exploratory and visual material prior to and continued along with the use of text materials. Likewise, ideas of size and shape, of time and distance,

and the many others that form a quantitative vocabulary are ideally suited to learning in context with their use in a normal situation.

Textbooks provide a valuable organization of arithmetic ideas which a teacher may use as her guide. One teacher may wish to begin a topic with reading and study from a text and then supplement the development of ideas and principles with exploratory and visual materials. Another may wish to introduce a topic with a discussion of situations and experiences and an exploratory-visual demonstration which leads to discovery of an idea, and to follow this with text materials after the basic understandings have been formed. Both approaches are valid. The artistry in teaching lies in knowing when and how to do the one in preference to the other. A good teachers' manual will give generalized guidance. For learning a process or a procedure such as division, a pattern to study and to use in reference is desirable. This may come from a book, or it may be presented on the chalkboard by the teacher. In any approach the discovery by a child or the explanations of the teacher concerning *why* and *how* remove the process from the mechanical stage.

Many pupils can become self-reliant and resourceful as they make discoveries in learning arithmetic. For example, a boy who discovers addition combinations can use several approaches to find the sums. To find "8 and 6" he can count objects, he can relate to known combinations or to doubles, or he can build to ten and the necessary amount beyond. Note the progressive maturity of these methods. This same boy may "undo" his addition into subtraction combinations and he may also frequently extend the methods to multiplication and division. For this type of learning, the guidance of the teacher is invaluable. Other pupils at later stages learn to interpret the textbook, and others find that they can reinforce ideas by consulting a textbook. This self-reliance is prized because it represents a method that is valuable in itself and can be used independently to extend any learning.

The greatest weakness of textbook use is really the fault of the teacher who forces memorization without understanding. A seventh-grade student illustrated the result of this unwise use of a text by his teacher when he asked another teacher for help

with "TTR2" problems. They were πr^2 exercises in which the symbol π had no meaning to the student and looked like two letter T's joined together.

In the use of a textbook the teacher must make a selection of practice exercises and problems. All pupils cannot and should not be expected to reach the same levels of understanding and of performance. The aim is to have each child extend and enrich his ideas as far as possible in a reasonable period of time. Flexible subgrouping in a class is therefore recommended. It is probably better for Joe to learn fewer things well than to have great uncertainty about many topics. For example, Joe can learn to add and subtract simple fractions like fourths and halves via the *cake pan* method and by drawing pictures. For him this is better than to try, but not succeed, in mastering the written symbols. On the other hand, Sam might well be expected to use his insight into number relationships to add and subtract in written symbols such fractions as thirds, fifths, and eighths.

The textbook is an invaluable instructional material. It provides an outline of ideas for the use of the teacher and children, it helps the teacher with the development of ideas, it provides practice exercises and problems for the pupils, and it is a useful reference book for them.

SUGGESTED INSTRUCTIONAL MATERIAL

A teacher who is selecting instructional materials for his classroom considers first the objective of teaching mathematics for his grade level and clearly outlines the ideas to be developed by the children. The outline of ideas may have major headings of number, operation, measurement, and relationship. With this outline in mind a teacher then considers methods of teaching the ideas and determines the materials needed.

The two sections which follow contain suggestions for instructional materials which teachers have found valuable to students learning arithmetic. The first section lists material for use on the primary level, the intermediate level, and the junior-high level to develop the ideas of number, operation, measurement, and relationship. Permanent equipment of a classroom may be selected

from these materials. This is followed by a list of sources of commercial materials and of supplies for the construction of teacher- or student-made material. The next section lists some films and filmstrips and their sources.

Primary Classroom

Number and Operation

1. Counters (cubes, buttons, tongue depressors)
2. Counting frames
3. Number cards and domino cards
4. Number chart (1-100)
5. Pocket place-value chart
6. Abacus (demonstration and individual)
7. Peg board (100 and 1000)
8. Number scale

Measurement

Instruments for the measurement of

1. Length: inch squares of paper, flexible ruler made of inch squares, ruler calibrated only in inches, ruler calibrated only in half inches, ruler calibrated only in quarter inches, flexible yardstick made of foot lengths, flexible yardstick made of inch lengths
2. Time: calendar, schoolroom clock, clock face with adjustable hands, mantle clock
3. Weight: simple teacher-made balance scales
4. Counting measures: egg cartons of different shapes, sections from egg crates
5. Money: play money (real money when available from school function)
6. Temperature: room thermometer, large adjustable model thermometer
7. Liquid measure: measuring cup; half-pint, pint, quart, half-gallon, and gallon milk containers
8. Dry measure: pint, quart, half-bushel, and bushel baskets
9. Geometric ideas: cardboard cut-outs of circles, triangles,

squares, and rectangles; objects in shape of cube, cylinder, sphere, and cone

Relationship

1. Counters
2. Fractional cut-outs
3. Paper for folding into fractional parts

Intermediate Classroom*Number and Operation*

1. Counters
2. Place-value pocket chart
3. Abacus
4. Peg board (100 and 1000)
5. Number scales: whole number, common fraction, and decimal fraction

Measurement

Instruments for the measurement of

1. Length: inch squares of paper, ruler calibrated in quarter inches, ruler calibrated in eighth inches, ruler calibrated in sixteenth inches, ruler calibrated in tenth inches, yardstick, odometer, tape measure, pedometer
2. Time: calendar, room clock, stop watch, sun dial, hourglass, timetables, standard time chart, time line
3. Weight: simple balance scales, nurses' scales, height-weight charts
4. Counting measures: egg cartons, sections from egg crates
5. Money: money from classroom functions, tokens, trading stamps
6. Temperature: classroom thermometer, clinical thermometer
7. Liquid measure: measuring spoons and cups; half-pint, pint, quart, half-gallon, and gallon milk containers
8. Dry measure: half-pint, pint, quart, half-bushel, and bushel baskets

9. Surface measure: inch square, foot square, and yard square cut-outs from newsprint; cross-ruled paper
10. Solid measure: cubic-inch wood cubes (more than 144), material for construction of a cubic foot and cubic yard
11. Geometric ideas: cardboard cut-outs of circle, triangle, square, and rectangle; objects in form of cube, sphere, cylinder, cone, rectangular prism

Relationship

1. Counters
2. Parts of egg cartons
3. Fractional cut-outs
4. Paper for folding and cutting ($8\frac{1}{2}$ " x 11" used notebook or typing paper cut in $8\frac{1}{2}$ " x 2" strips)
5. Tray of 100 blocks or chart of 100 squares
6. Fraction board
7. Number scale: whole number, common fraction, and decimal fraction
8. Materials using fractions: ruler, liquid measures, odometer, pedometer
9. Maps, diagrams, and other scale drawings
10. Tables of values from texts, newspapers, etc.
11. Graphs (bar, line, circle, and pictograph) from magazines, texts, etc.

Junior High School Classroom

Number and Operation

1. Counters
2. Abacus
3. Number scales: positive and negative whole numbers, common fractions, decimal fractions
4. Slide rule
5. Calculating machine

Measurement

Instruments for the measurement of

1. Length: foot rule, yardstick, meterstick, micrometer, odometer, pedometer, surveyor's tape, carpenter's rule
2. Surface measure: square-inch and square-centimeter ruled paper; lined peg board for outlining straight-line figures; newsprint cut in square inches, square foot, square yard, square centimeter, square meter; cardboard or wood cut-outs of many sized circles, squares, rectangles, trapezoids, parallelograms, and triangles
3. Volume measure: wood cubic inches and cubic centimeters; material for preparing cubic feet, yards, and meters; models of cubes, spheres, prisms, cylinders, cones, and pyramids; cylindrical and conic containers having equal diameters and altitudes; containers in shape of pyramid and prism having bases of the same size and shape and equal altitudes; instruments for measuring liquid and dry measure including measurement in cubic centimeters and liters
4. Temperature: Fahrenheit and centigrade scaled thermometers
5. Time: instruments for measuring time, standard time chart
6. Weight: instruments for measuring weight in pounds and in grams
7. Money: money from classroom activities, collection of foreign money
8. Others: hypsometer, gauges from old gas and electric meters, 100 squares (10 x 10) ruled on chalkboard, dowel rods to show formation of angles, adjustable triangles and quadrilaterals

Relationship

1. Fractional cut-outs
2. Tray of 100 blocks or chart of 100 squares
3. Fraction board
4. Number scales: positive and negative whole numbers, common fraction, decimal fraction, and percent

5. Maps, diagrams, blueprints, etc. (scale drawings)
6. Tables of values for reading and for construction of graphs
7. Graphs (pictograph, bar, line, and circle) for reading
8. Cross-ruled paper for drawing of graphs
9. Business forms from merchants, banks, building and loan companies, etc.
10. Pamphlets with information concerning bonds, stocks, social security, insurance, budgets, etc.
11. Information about tax rate, property valuation, etc., for city and county

Sources for Instructional Material

Many teaching materials are available commercially. A list of some of the companies which offer instructional devices and materials for their construction follows. Catalogs are available upon request.

Creative Playthings, Inc., Edinburg Road, Cranbury, New Jersey

Cuisenaire Co. of America, Inc., 246 East Forty-Sixth Street, New York 17, New York

Ginn and Co., Boston 17, Massachusetts

J. L. Hammett Company, Kendall Square, Cambridge 42, Massachusetts

Houghton Mifflin Company, 2 Park Street, Boston 7, Massachusetts.

Ideal School Supply Company, 8312 Birkhoff Avenue, Chicago 20, Illinois

Judy Company, 310 North Second Street, Minneapolis 1, Minnesota

W. M. Welch Scientific Company, 1515 Sedgwick Street, Chicago 10, Illinois

John C. Winston Co., 1010 Arch Street, Philadelphia 7, Pennsylvania

SUGGESTED FILMS AND FILMSTRIPS

The following films and filmstrips concerned with the social application and the historical development of mathematics have

been previewed by the authors and found useful for imparting information. (The addresses of distributors are given at the end of the list of films.)

Area and Volume. Six student participation filmstrips. Du Kane Corporation. Intermediate and junior-high grades.

Banks and Credit. Coronet Films. Junior-high grades.

Fred Meets a Bank. Coronet Films. Junior-high grades.

History of Measures. Six filmstrips on the history of area measure, the calendar, number system, telling time, linear measure, and weight and volume measure. Young American Films, Inc. Intermediate and junior-high grades.

History of Measurement of Length. Thrift. Is Seeing Always Believing? Business Methods for Young People. Filmstrips. Visual Education Consultants, Inc. Junior-high grades.

Installment Buying. Coronet Films. Junior-high grades.

Language of Graphs. Coronet Films. Intermediate and junior-high grades.

Let's Measure: Inches, Feet, and Yards. Coronet Films. Primary grades.

Let's Measure: Ounces, Pounds, and Tons. Coronet Films. Primary grades.

Let's Measure: Pints, Quarts, and Gallons. Coronet Films. Primary grades.

Man and Measure Series. Four filmstrips about early counting, measuring length, measuring time, and geometric figures. Filmstrip House. Primary, intermediate and junior-high grades.

Maps are Fun. Coronet Films. Intermediate and junior-high grades.

Measurement. Coronet Films. Intermediate and junior-high grades.

Measuring Time and Things. (Time and length.) Six student participation filmstrips. Du Kane Corporation. Primary and intermediate grades.

Money Lessons for Primary Grades. Filmstrip. Visual Education Consultants, Inc. Primary grades.

Number System. Encyclopedia Britannica Films, Inc. Intermediate grades.

Percent in Everyday Life. Coronet Films. Junior-high grades.

Story of Measurement. Filmstrip. Photo and Sound Production. Intermediate grades.

Story of Money. International Film Bureau, Inc. Junior-high grades. (Uses British money system as example of development of modern money.)

Story of Our Money System. Coronet Films. Intermediate and junior-high grades.

Story of Our Number System. Coronet Films. Intermediate and junior-high grades.

Story of Weights and Measures. Coronet Films. Intermediate and junior-high grades.

Weights and Measures. Encyclopedia Britannica Films, Inc. Junior-high grades.

What is Money? Coronet Films. Junior-high grades.

What Makes Us Tick? (New York Stock Exchange.) Modern Talking Pictures. Junior-high grades.

What Time Is It? Coronet Films. Kindergarten and primary grades.

Work of the Stock Exchange. Coronet Films. Junior-high grades.

Your Thrift Habits. Coronet Films. Intermediate and junior-high grades.

Distributors of Films and Filmstrips

Coronet Films, 65 East Water Street, Chicago 1, Illinois

Du Kane Corporation, St. Charles, Illinois

Encyclopedia Britannica Films, Inc., 1150 Wilmette Avenue.
Wilmette, Illinois

Filmstrip House, 347 Madison Avenue, New York 17, New York

- International Film Bureau, Inc., 57 East Jackson Boulevard, Chicago 4, Illinois
Modern Talking Pictures, 216 East Superior Street, Chicago 11, Illinois
Photo and Sound Production, 116 Natoma Street, San Francisco 5, California
Visual Education Consultants, Inc., 2066 Helena Street, Madison 4, Wisconsin
Young American Films, Inc., 330 West 42nd Street, New York 36, New York

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Definitions in Arithmetic

LEE E. BOYER, CHARLES BRUMFIEL,
and WILLIAM HIGGINS

THERE ARE two parts to the problem of formulating clear definitions of arithmetical concepts. First, we may focus our attention upon the number systems of arithmetic themselves. Within these number systems we note things that we designate as *numbers* (cardinal numbers, fractions, rational numbers), *operations* (addition, multiplication), *properties* (even, odd, divisibility) and *relations* (less than, greater than, square of). Clear and precise definitions for such concepts as those listed above have been formulated in recent years by mathematicians working in the foundations of mathematics. These definitions are not especially sophisticated or difficult to understand, but in general they have not yet been utilized fully in the literature pertaining to the teaching of arithmetic. One of the major objectives of this chapter is to make some of these modern definitions accessible to teachers of mathematics and writers of arithmetic textbooks.

Second, we are faced with an important pedagogical problem in the teaching of arithmetic. Many of the definitions of concepts



which have been formulated during the last century cannot be presented formally in the classroom. Consequently, a large vocabulary of helping words and phrases has been developed for pedagogical purposes. We must not forget, however, that the existence of this terminology can be justified only because it enables the student to show growth in dealing with quantities. All of this adjunct language of arithmetic instruction must be measured against the basic definitions. There is a tendency for language to develop beyond necessary terminal points and become unduly complicated. Many useless words have crept into the arithmetic vocabulary and some words are used incorrectly.

In the light of the foregoing, this chapter will be presented in two parts. Part I will deal with concepts that are currently used in arithmetic writing, in an effort to clarify their correct usage. Part II will deal with several fundamental arithmetic concepts as treated in so-called modern mathematics. The writers wish to

emphasize that beginning pupils will probably use the commoner language and organizational pattern of Part I. Both teachers and pupils should gradually become acquainted with and grow in the direction of the more general, and mathematically more useful, language pattern found in Part II.

A definition in elementary arithmetic should have the following characteristics.

1. A definition should contain only words previously defined or sufficiently primitive to be accepted as undefined.

2. A definition should be *correct*, but not necessarily *complete*. If it is not complete, it must be regarded as descriptive.

3. A definition must be useful. This is the most important criterion of all. Definitions are the tools of the mathematician. They are purposefully created to attack problems.

The following examples illustrate violations of the above principles.

Addition is the process of finding, without counting, the sum of two or more numbers. This violates principle 1, above. The word *sum* is undefined.

Zero is not a number; it is merely a place holder. This statement is incorrect; it violates principle 2. There is a confusion here between numbers, which are generally considered to be ideas, and numerals, which are symbols, often representatives of numbers. When you consider such symbols as 0, 3, 796, you look at physical things (blobs of printers ink in this case) which call to mind certain ideas that are numbers. In the same manner contemplation of the flag of one's country calls up certain abstract ideas. Certainly numerals, as physical objects, occupy positions in space just as the flag hangs on the flagpole. Numbers, as ideas, just as surely are not positioned in space.

Part I has two subdivisions. The first contains concepts that are comparatively well known and understood. The second lists concepts that are widely used in the field of measurement but are often used erroneously. Naturally, much more space needs to be given to define a concept in the second subdivision of Part I than in the first.

PART I

Concepts Generally Well Understood

ABSTRACTION

See GENERALIZATION, page 256.

ABSTRACT
NUMBER

This is a tautology. All numbers are abstract. See NUMBER, page 257.

ALGORITHM

Any mathematical procedure consisting of a number of steps, each step applying to the result of the one preceding it. Between the tenth and fifteenth centuries *algorithm* was synonymous with positional numeration and was an emphasis on form or pattern of operation. The body of mathematical knowledge we now call arithmetic was called algorism for centuries probably because the people who used algorithms to calculate were called *algorists* in contrast to the *abacists* who used the abacus to calculate. Today, it seems, algorithm and algorism are synonymous. The *Mathematics Dictionary* by James and James lists *Algorithm*, not *Algorism*. Examples of algorithms are:

(a) $\begin{array}{r} 734 \\ -158 \\ \hline 576 \end{array}$	(b) $\begin{array}{r} \sqrt{5625} / 75 \\ 49 \\ 145 \overline{)725} \\ \underline{725} \end{array}$	(c) $\begin{array}{r} 721 \\ \times 301 \\ \hline 721 \\ 2163 \\ \hline 217021 \end{array}$
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APPROXIMATE
NUMBER

Term applied to a number established by a process of measuring. This is, of course, an abuse of language, as is also concrete number. Examples: 12 miles, 15 feet, 16.3 inches.

BASIC FACT

A fundamental operation in arithmetic involving two one-place numbers with the answer. Examples:

$$\begin{array}{l} \text{In addition, } 2 + 3 = 5 \\ \text{or } 6 + 7 = 13 \end{array}$$

In subtraction, $3 - 3 = 0$
or $15 - 7 = 8$

In multiplication, $4 \times 0 = 0$
or $7 \times 2 = 14$

In division, $2 \div 2 = 1$
or $36 \div 4 = 9$

(It is not possible to use zero as a divisor.)

See also NUMBER COMBINATION, page 257.

BINARY NUMERALS

Numerals whose digits have positional values that are powers of two. Thus the binary numeral 100111 has a decimal value of thirty-nine. This can be seen from the following:

$$100111_{(2)} = 1(2^5) + 0(2^4) + 0(2^3) + 1(2^2) \\ + 1(2^1) + 1(2^0) = 39_{(10)}$$

CANCELLATION

The use of the word *cancel* is not recommended. Its use seems to encourage promiscuous crossing-out of numerals. In the example below, we might think, "after dividing 7 and 14 by 7, and then multiplying, we have $\frac{1}{7}$ as an answer." The term *reduction* is preferred to the term *cancellation*.

$$\frac{3}{14} \times \frac{7}{5} = \frac{3 \times \cancel{7}^1}{2\cancel{14} \times 5} = \frac{3}{10}$$

CIRCLE

Although this term is frequently applied both to the surface of a plane bounded by a closed curve and to the closed curve itself, in mathematics it applies only to the curve itself. Thus, a circle with center at A and radius r is the set of all points X in the plane such that the length of AX is r .

CONCEPT

A concept is a *set*. By the process of generalization a child discovers a characteristic

common to all members of some set. When this discovery has been made we say the child understands the concept. Thus the concept of *seven* is the idea of size or quantity common to all groups of seven such as seven fingers, seven steps, seven stars, seven inches, seven minutes, seven words, etc. The concept is the thing understood. The generalization is the thought process by which the concept was discovered.

CONCRETE NUMBER

By virtue of the definition of a number it is impossible to have a concrete number. A number may, of course, characterize the size of a group of concrete things, such as four dogs.

DECADE NUMBER

Numbers from 0 through 9 are said to be in the first decade, numbers from 10 through 19 are in the second decade, and so on. When numbers are added, as in the example

$$\begin{array}{r} 9 \\ 6 \\ 8 \\ +9 \\ \hline \end{array}$$

it is necessary to add by endings and note that the sum moves from one decade to another. For instance, adding 9 and 6 yields a sum in the second decade. Adding an 8 to this unseen number, 15, yields a sum in the third decade. The final sum, 32, is a fourth decade number.

DECIMAL

As a single word it is slang for *decimal fraction*. Decimal fractions always have denominators which are powers of 10, and may be written as .3, .15, .125. The power of 10 in the denominator is numerically equal to the number of decimal places in the decimal fraction.

DIGIT	Any one of the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. The numeral 203 contains the digits 2, 0, and 3.
DRILL	Frequent, formal, and speedy repetitions engaged in to fixate learning. Emphasis is on frequent repetition with practically no effort to gain meaning or understanding. In recent years educational psychology has provided principles whereby drill may be more interesting and effective. To the extent that these principles are applied, use of the word <i>drill</i> has given way to use of the word <i>practice</i> .
ENUMERATION	The process of assigning the number names to the individual objects in a group in order to find the total number.
EQUAL OR EQUALS	<p>Equal is a verb used with plural subjects; equals, with singular subjects. Numerals representing abstractions are all singular. Thus the numerals 1 and 5 are both singular. Numerals representing or associated with concrete objects are singular if one is involved and plural if more than one is involved. The words <i>plus</i> and <i>minus</i> are used as prepositions; <i>and</i> is always a conjunction giving rise to a compound or plural subject. The following illustrations help clarify correct usage:</p> <p>5 + 2 equals 7. 7 minus 2 equals 5. 2 multiplied by 6 equals 12. 8 + 4 equals 12. 5 bears multiplied by 2 equal 10 bears. 1 bear and 5 bears equal 6 bears. 1 and 2 equal 3. 1 child plus 6 children equals 7 children. 7 mice — 6 mice are 1 mouse.</p>

1 plate divided by 4 equals $\frac{1}{4}$ plate.

4 quarts equal 1 gallon.

EXACT NUMBER

Term applied to a number established by the process of counting. Examples: 8 cups, 5 birds, 123 cars. See APPROXIMATE NUMBER.

EXPLORATORY MATERIAL

Materials that can be moved or touched. They should be the means of enabling a pupil to discover a principle or fact that he would not discover or understand without the use of that material.

FACTOR

A factor of a number is one of two or more whole numbers which have a product equal to the given number.

FIGURE

This term is not preferred as a name for a number symbol. See DIGIT and NUMERAL.

GENERALIZATION

A statement that is true for every member of a set. The application of a generalization in subtraction may be illustrated by means of the following exercises:

$$\begin{array}{r} \text{(a)} \quad \begin{array}{r} 6\overline{3}5 \\ -1\overline{4}2 \\ \hline 4\overline{9}3 \end{array} \quad \text{(b)} \quad \begin{array}{r} 7\overline{8}3 \\ -2\overline{8}7 \\ \hline 4\overline{9}6 \end{array} \quad \text{(c)} \quad \begin{array}{r} \overline{124}6 \\ -\overline{25}3 \\ \hline \overline{99}3 \end{array}$$

The generalization is: If a digit in the subtrahend exceeds the corresponding digit in the minuend by unity (after regrouping), the remainder is always nine. Another example: The product of two odd numbers is an odd number.

INSIGHT

Keen discernment and comprehension of the fusion of mathematical and social phases of arithmetic in the mind of the learner.

LEAST COMMON DENOMINATOR

The least number which is divisible by the denominators of a set of *arithmetic frac-*

tions. Thus, the least common denominator of $\frac{1}{3}$, $\frac{1}{4}$ and $\frac{1}{6}$ is 24 because this number is less than other possible common denominators such as 48 or 96. The least common denominator is also called the *least common multiple*.

MEANING
PROGRAM

A program that emphasizes exploration and discovery in problem solving and aims at acquiring mathematical understanding.

MEASURE

To measure a quantity means to find how many times it contains a standard quantity of the same kind, that is, a commonly accepted unit of measure.

NUMBER

There is no simple definition for *number*. See Part II for definitions of particular sets of numbers.

NUMBER BASE

The base of a system of numeration is the number of units in a given digit's place which must be grouped to have a value equal to one of the next higher place in the system. The number of digits used in a system of numeration is equal to the base of that system.

NUMBER
COMBINATION

$7 + 2 = 9$ is sometimes called a *number combination* and sometimes a *number fact*. Preferred usage would have $7 + 2 = 9$ and $2 + 7 = 9$ considered as the same addition combination but two different addition facts.

NUMBER
GROUPING

This idea is basic to the decimal system of numeration in that it recognizes certain special groups whose sizes are powers of 10. That is, ones (10^0), tens (10^1), hundreds (10^2), thousands (10^3), etc. Another use of

this idea is found in addition when a child recognizes in an addition problem such as

$$\begin{array}{r} 8 \\ 5 \\ 5 \\ 2 \\ \hline \end{array}$$

that 8 and 2 are 10, two 5's are 10 and thus the final sum is 20.

**NUMERAL vs.
NUMBER**

A numeral is a symbol which represents a number. Examples: 1, 2½, 304. A synonym in common, although incorrect, usage is *number*. A numeral is a name for a number. The numerals 7 and VII are names for the number seven just as John and Johnny are names for the boy John.

**PRACTICE
PROBLEM**

See DRILL, page 255.

A quantitative situation which cannot be solved by a habitual response. A quantitative situation described in words in which a definite question is raised, but for which the arithmetical operation is not indicated. Any arithmetical situation involving doubt and difficulty and an unknown answer. Its characteristic feature is doubt rather than difficulty or an unknown answer.

**PROBLEM
SITUATION**

A situation presented to the child that challenges him and requires thinking through a situation to a solution. Not all mathematical situations are problem situations for all children. A child may already know a solution and therefore needs no thinking to arrive at a solution. A less mature child may not even understand what is wanted and will therefore be unable to solve the problem at all; thus for him it is not a problem situation.

QUANTITATIVE THINKING	Thinking in situations in which a consideration of amount is essential.
READINESS	The background skills and concepts essential to understanding the arithmetical process to be developed. The attainment of readiness is a continuing process of becoming more ready than one was previously.
RECIPROCAL	If the product of two numbers is unity, either number is the reciprocal of the other. Examples: 3 and $\frac{1}{3}$, $\frac{2}{3}$ and $\frac{3}{2}$, and $\frac{1}{x}$ and x are three sets of reciprocals. The reciprocal of a non-zero number is 1 divided by that number.
TEEN NUMBERS	The numbers 10 to 19 inclusive. They are so named because they constitute the sums of 10 added to the numbers 0 to 9 inclusive.
ZERO	The <i>number</i> zero, which is a concept, is often confused with the <i>symbol</i> 0 which is a numeral. Zero marks the point on the real number scale which separates the positive numbers from the negative numbers. Thus, zero is an integer but it is neither positive nor negative. It is not good usage to call zero "nothing," "naught," or "cipher."

Concepts Used in Measurement That Frequently Are Used Erroneously

Every measurement is an approximation. Because of this fact numbers resulting from measuring processes and written as 15 feet or 6.3 inches are called approximate numbers (see page 252). A measurement recorded as 15 feet represents, by common agreement, any length between 14.5 feet and 15.5 feet. Similarly, 6.3 inches represents any length between 6.25 inches and 6.35 inches. The significance of the approximate character of these numbers becomes evident if one attempts to determine the number of square feet in a flower bed said to be 15 feet square or the number of square inches in a tile 6.3 inches square. In the first case the

area may range between 210.25 square feet $(14.5)^2$ and 240.25 square feet $(15.5)^2$; in the second case, between 39.0625 square inches and 40.3225 square inches. Such great variations may give rise to disappointments, and even failures, when close calculations are involved if the possible tolerances are overlooked. To express the tolerance concepts involved when calculations are made using approximate numbers, definitions for the following terms are helpful: ACCURATE OR ACCURACY, CORRECT, PRECISE OR PRECISION, POSSIBLE ERROR, RELATIVE ERROR, and SIGNIFICANT DIGITS. Since these terms are only six in number, they will be discussed in the order of mathematical need rather than in alphabetic order.

**SIGNIFICANT
DIGITS**

The term *significant* is here used in a technical sense. The reader must not assume that a non-significant zero has no importance, role, or function. The following five statements and accompanying examples will serve to guide students in recognizing digits that are significant and those that are not.

1. All of the non-zero digits are always significant wherever used. Examples: 1, 2, 3, 4, 5, 6, 7, 8, and 9 are always significant.
2. Zeros occurring between non-zero digits are always significant. Examples: 201; 80,605; 19,008; 80.005.
3. Final zeros of an approximate whole number may be significant. Example: If a length recorded as 3000 feet were measured to the nearest thousand feet, only the digit 3 would be significant; if the length were measured to the nearest hundred feet, the zero to the right of the 3 would also be significant; if the length were measured to the nearest ten feet, the two zeros to the right of the 3 would be significant and if the length were measured to the nearest foot all of the zeros would be significant. In common practice the final zeros of an approximate whole number are regarded as non-significant unless the nu-

meral is accompanied by descriptive adjectives stating that the measurement was made to the nearest unit of some denomination represented in the numeral by a zero. Examples: (a) 4,000 would be regarded as having three non-significant zeros, (b) 4,000 to the nearest ten would be regarded as having only one non-significant zero—the one farthest to the right, (c) 1,067,000 to the nearest hundred would be regarded as having two non-significant zeros—the two farthest to the right.

4. Final zeros of an approximate number expressed as a decimal fraction are always significant. Examples: (a) 63.0, (b) 10.20, (c) 9.0200 and (d) 60.0.
5. Zeros used in decimal fractions merely to locate the decimal point are never significant. Examples: (a) 0.03, (b) 0.0043, and (c) 0.00286.

There are five significant digits in each of the following numerals: 302.06, 0.0072689, 500.00 and 72,346. The numeral 93,000,000 miles, stating the approximate distance from the earth to the sun to the nearest million miles, has only two significant digits.

POSSIBLE ERROR Since every measurement is approximate, every number representing a measurement must of necessity have some error. This error may not be greater than 0.5 of the unit in terms of which the number is expressed. Thus, the possible error of 12.5 in. may range from 0.00 to 0.05 in. since this number represents a quantity between 12.45 in. and 12.55 in. In practice *possible error* has come to mean *greatest possible error*. In the case of 12.5 in. the possible error is 0.05 in. The measurements 2.12, 8.610, and

0.1245 have the possible errors of 0.005, 0.0005 and 0.00005 respectively.

RELATIVE ERROR The quotient derived by dividing the possible error of a measurement by the measurement itself is called the relative error of the measurement. Examples: (a) The relative error of 2.4 is $\frac{0.05}{2.4} = \frac{1}{48}$; (b) the relative error of 93,000,000 miles is $\frac{500,000}{93,000,000} = \frac{1}{186}$.

CORRECT Correct implies freedom from fault or error as judged by some accepted standard. If a result is achieved by accepted procedures it is correct. Thus, $6 + 8 = 14$ and $7 \times 9 = 63$ are correct if we have accepted a decimal system of numeration. The results are incorrect if we have accepted a system whose base is 12; in this case the correct results would be 12 and 53 respectively. A number is said to be used correctly if its possible error is less than 0.5 of the unit in which the number is expressed. Thus, 45 feet is used correctly if it represents a distance between 44.5 feet and 45.5 feet.

ACCURATE, OR ACCURACY The accuracy of an approximate number depends upon its relative error. Since the relative errors of 3 feet, 0.3 foot and 0.03 foot are all equal to $\frac{1}{10}$, the measurements 3 feet, 0.3 foot and 0.03 foot are all equally accurate. Similarly, 24 inches, 2.4 inches and 0.24 inch are equally accurate because $\frac{1}{10}$ of each of these measurements may be error.

It should be noticed that the numbers in each of these two sets have the same number of significant digits. Each number in the first set had one significant digit and each number in the second set had two significant digits.

In common practice, all measurements that

have the same number of significant digits are said to have *about the same degree of accuracy*. Examples: 8154, 90.57, 6.056 and 0.7500 are said to have about the same degree of accuracy because all of them have four significant digits.

The concept of accuracy is basic when the operation of multiplication or division is performed using approximate numbers. Neither the product nor the quotient may be more accurate than the least accurate number involved in the operation.

PRECISE OR PRECISION

The precision of an approximate number depends upon the size of the unit which it represents. Examples: The expression *4 inches* is more precise than the expression *4 feet*. Similarly, *24.31 feet* is more precise than *24.3 feet*.

The concept of precision is basic when the operation of addition or subtraction is performed using approximate numbers. Neither the sum nor the remainder may be more precise than the least precise number involved in the operation. If the approximate remainders (d), (e), and (f) are given as the results to subtraction exercises (a), (b), and (c),

(a) 62	(b) 7.2	(c) .62
$\begin{array}{r} 47 \\ \hline 15 \end{array}$ (d)	$\begin{array}{r} 4.7 \\ \hline 1.5 \end{array}$ (e)	$\begin{array}{r} .47 \\ \hline .15 \end{array}$ (f)

(d), (e), and (f) are equally accurate because the relative error of each is $\frac{1}{36}$; (d) and (f) are correct; (e) is incorrect. Each numeral of (d), (e), and (f) has a different precision; (e) is more precise than (d) because (e) is expressed to the nearest tenth of a unit and (d) is expressed only to the nearest unit. Since (f) is expressed to the nearest hundredth of a unit it is more precise than (e).

PART II

We shall discuss briefly a few of the primitive undefined terms needed for subsequent definitions. Basic for our purpose is the concept of a *set* (collection, class) of objects. We make no attempt to define the set concept. It seems intuitively clear that the mind has the power to consider a collection of distinguishable objects as a single entity. The elements of a set may be physical objects or abstract ideas. There is no objection if a person chooses to construct a set consisting of Julius Caesar, the polar star and the number 25. It is difficult to imagine any reason for constructing such a set. It does not appear to be inherently useful. As we proceed it will become apparent that the definitions of arithmetic are closely associated with the construction of useful sets.

Some pairs of sets may be placed in *one-to-one correspondence*. For example, the five fingers on one hand may be matched one-to-one with the five fingers on a second hand. If all the seats in your classroom are filled with one student in each seat and no students standing, then we say that there is a one-to-one correspondence between pupils and seats. Any two sets that may be placed in one-to-one correspondence are said to be *equivalent* to each other.

It is useful to think of a *cardinal number* as a family of sets. Bend down the thumb on one hand. The remaining set of fingers can be placed in one-to-one correspondence with any quartet, any set of quadruplets or any complete table of bridge players. Form in your mind the family consisting of all sets which can be put in one-to-one correspondence with this set of fingers. This family we call cardinal number four. It is quite correct to say of the set of starting players on a basketball team that *this set belongs to the cardinal number five*. That is, the cardinal number 5 is a great family of sets which includes this special set of basketball players.

The important usages of number in elementary arithmetic involve cardinal numbers. Ordinal numbers play a minor role, but we shall discuss the ordinal concept briefly. It is usually said that first, second, third, etc. are ordinal numbers. This is a meaningless remark. Certainly it is not meant that the words themselves, *first*, *second*, are ordinals. It would seem to be implied that these words are names for ordinal numbers in the same manner that the symbol *cow* is a name for an animal. The word *cow* does not name

a particular animal. The fundamental question must still be asked: *If first is the name of an ordinal number then what is this ordinal number named by the word first?*

A modern definition of ordinal number may be made intuitively acceptable by examining closely the images evoked in one's mind by a statement like: *He was third in line*. A man could not be third unless there was also a second and a first. The naming of any ordinal immediately focuses our attention upon the set of preceding ordinals. A definition of ordinal number arises from this intuitive fact.

When we speak of page one, or the first student in a class, we visualize the empty set. For the set of students ranking higher than the first student is indeed the empty set. Our formal definitions now begin.

Ordinal zero is the empty set.

Ordinal one is the set whose sole element is ordinal zero, the empty set.

Ordinal two is the set containing the two elements, ordinal one and ordinal zero.

It is obvious how this definition proceeds. Ordinal ten is a set which belongs to the cardinal ten. It contains ten elements, namely the ordinals 9, 8, 7, 6, 5, 4, 3, 2, 1, 0. *Each ordinal is the set of all preceding ordinals.*

The following sequence of definitions employs primitive undefined terms like *set*, *member of a set*, and *cardinal number*. We designate sets by bold type capital Roman letters, **S**, **T**, ... and the cardinals of these sets by bold type small Roman letters **s**, **t**, ... If **S** is the set of letters in the word *clod* then **s**, the cardinal number of this set, is 4. If **S** is the set of live lions in your elementary classroom (the null set) then **s**, the cardinal number of this set, is 0.

1. **DISJOINT SETS.** Two sets **S** and **T** are said to be disjoint if there is no object in either set which is a member of both sets.

2. **UNION OF TWO SETS.** If **S** and **T** are any two sets then the set consisting of all those objects and only those which are in at least one (possibly both) of the sets is called the union of **S** and **T**. We designate this new set by **S U T**.

For example, the union of two sets, the one consisting of three fingers on one hand and the other consisting of all five fingers on the same hand, is simply the set of five fingers.

We may now formulate a precise definition for addition of cardinal numbers.

3. SUM OF TWO CARDINAL NUMBERS. If s and t are any two cardinal numbers we define a cardinal number $s + t$ called the sum of s and t . Choose any set S having the cardinal number s and then choose a set T having cardinal number t such that S and T are disjoint. Form the union $S \cup T$. We define the cardinal number $s + t$ to be the cardinal number of this new set.

Note the importance of stipulating that the sets S and T be disjoint. Without this requirement $3 + 5$ could be 5, 6, 7, or 8, depending upon the choice of sets.

Children sometimes add by illustrating a union of sets. The child asked to add 3 and 4 may form a set of 3 fingers on one hand, 4 fingers on the other and bring these sets together, forming the union.

Let us relate the definitions of *union of sets* and *sum of cardinals* to an ancient pedagogical problem. The child is often told that he cannot add unlike things. This assertion seems to raise logical difficulties for both children and teachers. Notice that when we distinguish sharply between forming the union of two sets and adding two cardinal numbers there is no confusion. The mind is free to form a set consisting of four watermelons and three pigs. This is not adding watermelons and pigs. This is simply using the mind as a tool to contemplate these seven objects as members of one set. To say that the mind is not free to form such arbitrary groupings of objects into sets is ridiculous. We *add* cardinal numbers. We *form the union* of sets. These are separate but related concepts. We may, if we wish, add the cardinal numbers three and four by thinking of sets of three pigs and four watermelons and recognizing that the union of these sets belongs to the cardinal number seven. This point of view removes all mystery from the statement, meaningless to many children, that only like things may be added.

4. PRODUCT OF TWO SETS. The product of two sets $S \times T$ is the set of all possible ordered pairs such that the first element in

each ordered pair belongs to **S** and the second to **T**. The ordered pair (a, b) is a different element in the set from the ordered pair (b, a) .

For example, let **S** consist of a ham sandwich and a cheeseburger. Let **T** consist of a glass of milk. Then $\mathbf{S} \times \mathbf{T}$ contains two ordered pairs. Each pair has for its first element a sandwich and for its second the glass of milk. Notice that the set $\mathbf{T} \times \mathbf{S}$ is conceptually distinct from the set $\mathbf{S} \times \mathbf{T}$. Of course this definition of set multiplication is constructed in order to define multiplication of cardinal numbers.

The reader is undoubtedly familiar with the definition of multiplication of cardinal numbers which relates multiplication to repeated addition. For example, $3 \times 4 = 4 + 4 + 4$ while $4 \times 3 = 3 + 3 + 3 + 3$. It is clear that when multiplication is defined in this way, many special products must be defined separately. For it is without meaning to define 1×4 as the sum of one four if the word *sum* presupposes the existence of two or more addends. It is possible to avoid this difficulty by defining 1×4 to mean $4 \times 1 = 1 + 1 + 1 + 1$. But even this trick breaks down for the four products 1×1 , 1×0 , 0×1 , 0×0 . When a definition of multiplication is based upon addition then these four products must be separately defined.

The reader will note that the definition given below frees multiplication of all dependence upon addition. This definition has the virtue of being completely general. No special products need be considered separately.

5. PRODUCT OF TWO CARDINAL NUMBERS. If s and t are any two cardinals, we define a cardinal $s \times t$ and call it the product of s and t . Let **S** and **T** be any two sets belonging to the cardinals s and t respectively. We form the product set $\mathbf{S} \times \mathbf{T}$ and define $s \times t$ to be the cardinal number of this product set.

One can relate this definition to simple physical situations. If we ask how many ways 4 boys and 3 girls can form mixed couples for dancing, we are led to consider a product set of 12 elements.

Many teachers find confusing the discussion of multiplication which asserts that the multiplicand must be a concrete number and the multiplier an abstract number. In physics classes students will be taught to write " $12 \text{ ft.} \times 5 \text{ lb.} = 60 \text{ ft. lb.}$ " This

appears to be a violation of the above multiplication principle. However, if we distinguish sharply between the two concepts, *product of two sets* and *product of two cardinal numbers*, everything is quite clear. We see at once that any two sets may be multiplied. If you wish to choose a set S consisting of three gentleman bears and a set T consisting of five lady elephants and then form the set product $S \times T$ you are certainly free to do so. Certainly we can imagine the bears and elephants dancing together and visualize the 3×5 ordered pairs of bear-elephant dancers. The formation of the product of two sets is a tool that we use to define the product of two cardinal numbers.

6. FRACTIONS. Fractions are ordered pairs of cardinal numbers for which operations called addition and multiplication are defined. We omit the familiar definitions for these operations. The first number in the ordered pair is called the numerator and the second is called the denominator. The denominator may not be the cardinal number zero.

This definition tells what fractions are. It is not a compilation of the uses of fractions. A little reflection should convince the reader that in every physical situation in which a student encounters the fraction $\frac{2}{3}$ he is confronted by two sets, the one of cardinal number 2 and the other of cardinal number 3.

There is a difference between sending presents to two out of a set of three friends and sending presents to twenty out of a group of thirty friends. The fractions $\frac{2}{3}$ and $\frac{20}{30}$ are not the same. But these fractions certainly have some common property. We may say that they are *equivalent* to each other. This concept, *equivalence of fractions*, may be called to the attention of children in many ways. For example, in the sharing of music books, one to every two children, it is seen that there are also two books for every four children, three books for every six children, etc. Thus the concept of the set of fractions

$$\left\{ \frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \dots \right\}$$

arises naturally.

7. RATIONAL NUMBERS. The set of all fractions equivalent to one fraction is called a rational number.

For example, every fraction in the set $\{\frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \frac{4}{16}, \dots\}$ is equivalent to the fraction $\frac{1}{4}$. The words *rational number one-fourth*, *rational number two-eighths* and the symbols $\frac{1}{4}$, $\frac{2}{8}$, etc. may be considered to be merely different names for this one set.

The basic operations, addition and multiplication of rational numbers, are of course defined in terms of addition and multiplication of fractions. Let us add the rational numbers $\frac{1}{4}$ and $\frac{1}{4}$.

The rational number $\frac{1}{4}$ is the set of fractions

$$\{\frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \frac{4}{16}, \frac{5}{20}, \dots\}$$

Rational number $\frac{1}{4}$ is the set

$$\{\frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \frac{4}{16}, \frac{5}{20}, \dots\}$$

Pick any two fractions, one from each set. Add these fractions. Determine the rational number which contains this sum. This rational number

$$\{\frac{7}{12}, \frac{11}{12}, \frac{13}{12}, \frac{17}{12}, \dots\}$$

is the sum of rational number $\frac{1}{4}$ and rational number $\frac{1}{4}$.

Multiplication of rationals may be treated in a similar manner.

Our basic objective is now accomplished. We have presented a chain of modern definitions which form the foundation for much work in arithmetic. Definitions of this type have been formulated for all mathematical concepts. Every mathematical concept, not only in arithmetic but also in advanced mathematics, is a clearly defined set of objects constructed by the mind of man. Relations like *equality* and *greater than*, operations like *addition* and *multiplication*, properties like *even* and *odd* may be defined precisely by constructing sets which we designate as these concepts. The chief advantage of this set theoretic approach is that haziness and vagueness are removed from the language of arithmetic.

It goes without saying that the teacher who understands these definitions will not present them formally to children. However, it is of great importance that the teacher present descriptions of arithmetical concepts in such a way that the child who continues to study mathematics will not absorb erroneous concepts which will need to be unlearned.

Modern Mathematics and School Arithmetic

ROBERT L. SWAIN

MANY TEACHERS, as they read in newspapers and magazines or hear from administrators and speakers at conferences that the mathematics curriculum must be *modernized*, feel uneasy about the unknown prospect. They want to know, "What is this modern mathematics? Will we have to go back to school to learn it? How can you change arithmetic anyway—facts are facts, and $2 + 2 = 4$ is here to stay! Is the modern movement just a passing fad? Is it something dreamed up by university scholars who have never worked with children? Will we be pressed to teach material that is too hard or otherwise unsuitable for our young pupils?"

We propose to deal with these and similar questions by discussing ways in which modern mathematical attitudes may influence future school programs and by citing some illustrations of modern approaches to arithmetical topics. For an expanded treatment of some of the topics discussed in this article, see the school text

series (2) and (6) and the author's text for elementary teachers (9).

Speaking before the 1957 annual meeting of the National Council of Teachers of Mathematics, Robert E. K. Rourke (7) cited several reasons for drawing upon modern mathematics in building a high school curriculum: to *broaden old ideas* and *introduce important new ideas*, to *clarify*, *simplify*, and *unify*. These goals are also among the ones we must keep in mind while we think about the elementary curriculum.

We wish of any new program that it offer pupils a better chance of acquiring clear understandings than they had under the old. Such a program may not make the teacher's task easier, but it is certain to make it more satisfying and often exciting. To be sure, this theme of meaningful learning is not new; it has been with us since about 1935. The resources and means now becoming available, perhaps from modern psychology as well as from modern mathematics, to improve the effectiveness of the classroom approach are new.

It is especially important that the pupil's attention be constantly directed toward the concrete bases of mathematical concepts and methods. We shall dwell upon this theme, as well as upon the general theme of meaning, or understanding, in all that follows.

Finally, we wish of a new program that it return the course of elementary school mathematics to the mathematical mainstream. Most textbooks now in use display many definitions and concept formulations, terms and notations, and even computational procedures that are out of date, wrong, misleading, or otherwise unsuitable. As students continue their educational pursuits, they have to unlearn as they learn. It is as though children were made to learn a special kiddie English from primers filled with poor grammar, awkward diction, and baby talk, supplemented by dictionaries loaded with misspellings and faulty definitions. The eighth grade child ought to have acquired an arithmetical background that will serve him adequately in the usual routine of life and work, yet which will provide a sound foundation for further training.

THE MEANING OF "MEANING"

Nearly all educational leaders and writers on arithmetic have climbed upon the bandwagon labeled "Meaning," but it is often no more than lip service that they render. *Meaningful learning* becomes for them a catch-all phrase useful for justifying whatever pedagogical procedure they may advocate.

Probably no satisfactory capsule definition of *meaning* can be contrived. However, we can single out a few significant characteristics and illustrate with examples to point up the distinction between a genuine and a counterfeit approach.

In a program that stresses meaning, pupils should be guided toward the development of structural insight. This is almost too broad a term to be useful, but it implies an attitude or point of view or orientation in which there is a sensitivity to pattern and to law, and an appreciation of connections or interrelations between concepts and between methods and techniques. A child, for example, ought not to view addition and subtraction as unrelated operations. He should know that subtracting 2 reverses the effect of adding 2. And he should know that adding 2 corresponds not only to putting two more objects into a given group, but also to counting forward two more from a given number—with subtraction corresponding to counting backward. He should be able to retrieve a forgotten number fact, like $9 + 7 = 16$, in several ways: by beginning at nine and counting seven more; by taking one from the seven to go with the nine to make ten, leaving six; by recognizing that nine is one more than eight, and seven one less, that is, $9 + 7 = 8 + 8$; etc.

The child must thoroughly grasp the place value structure of our numerals before he can understand any standard computational process. It is only parroting a rule to say that two times 34 is 68 *because* two times four is eight and two times three is six. The explanation must pierce more deeply. It must take into account the composition of the number 34 as the sum of three tens and four units (ones). And it must ultimately make use in some way of the distributive law—to multiply a sum by a number you must multiply each term of the sum by the number, then add. (In the early grades, of course, the teacher need not explicitly

refer to such a general law, but may appeal to the child's intuitive grasp of the relationship involved. The intuition must itself be bolstered by concrete manipulations with objects like blocks, coins or toothpicks, and with devices like number pockets or the open-end abacus.) The development of structural insight can be encouraged by such means as having children estimate approximate answers to problems before they work them, do mental arithmetic, and discuss problems without numbers.

In a program that stresses meaning, there should be constant and spirited interplay between the general and the particular—and also between the abstract and the concrete. The teacher who is told in her early training to go from particular to general and from concrete to abstract often learns the lesson too well, always thereafter obeying the one-way signs thus implanted in her mind. Actually, much learning takes place the other way about. A two-year-old may point to every man as a *daddy*, then later learn to particularize his reference. The important principle to be borne in mind with respect to the general and the particular is not that one precedes the other, but that they supplement and reinforce each other. When a child learns what the numerals 10, 11, 12, . . . , mean in terms of tens and ones, the place value idea (for two digits) begins to take shape in his mind, and this development illuminates further examples for him. Thus before being told about 17, he might guess that it means one ten and seven ones, and find his guess verified; at the same time, he might fail to guess what 34 means. The particular and the general interplay in his mind until the general concept is thoroughly implanted, but the interplay and reinforcement still continue, as the child makes frequent application of the concept. *General* and *particular* are actually relative terms rather than true opposites; similarly, *abstract* and *concrete*.

As another illustration, consider the recognition that $2 + 3$ and $3 + 2$ name the same number. A child might discover this by counting on his fingers: 1—2—1—2—3 for $2 + 3$, and 1—2—3—1—2 for $3 + 2$. In each case he finds he has counted all the fingers of one hand. A complete, in-sequence count then gives 1—2—3—4—5. So $2 + 3 = 3 + 2 = 5$. The child may also discover that $4 + 2$ and $2 + 4$ name the same number, also $5 + 7$

and $7 + 5$, and so on. He should soon grow aware of an emerging pattern: the relationship that we call the commutative law (or rule of commutation) of addition. This states that in adding, the order of the terms may be changed without affecting the result.

To understand the commutative law, however, it is not enough that the child should simply sense the pattern. Nor is it enough that he should check it completely in all possible cases of adding two digits. Logically, a complete check constitutes a proof. But for understanding, more than just a formal proof is needed. The commutative law is a structural property of addition. To see its *why*, we must reveal the structure. For the case $2 + 3 = 3 + 2$, take two marbles in your left hand, three in your right. Put the three with the two. This illustrates $2 + 3$. Combining the groups the other way around illustrates $3 + 2$. The basic general idea is apparent: When two piles are brought together into one, the order of combining does not matter.

Once grasped, the commutative law of addition becomes both an aid to understanding and a useful tool in numerical manipulation. There is a similar commutative law of multiplication. We have already mentioned the distributive law. There are also associative laws of addition and multiplication, which allow freedom in grouping terms.

The five laws named above are the *basic laws of arithmetic*. Their patterns recur again and again in computational work. Like other mathematical laws, they are specific and precise (i.e., their area of application is limited in some definite way). They must be heeded literally. While analogy is an important tool for aiding discovery, it is not a sound basis for proof. Thus $2 \times 4 = 4 \times 2$, but the analogous relation $2 \div 4 = 4 \div 2$ is false. There is a commutative law of multiplication, but not one of division. A more subtle illustration of the traps that await the unwary is offered by a journal article published in 1957. The writer cites a "rule of likeness" which she applies in three different situations: (1) to derive a similar fallacious rule that "only like quantities can be added," (2) to justify the vertical aligning of decimal points in adding, (3) to explain why we must get a common denominator when adding fractions. This writer's fallacy is not only a very common one, but one which completely shuts the door to mean-

ing and understanding. Her "rule" is no more than recognition of the obvious fact that we can find likenesses within many essentially dissimilar processes. But in itself, to find a partial likeness contributes nothing to the explanation of the processes. Both a pencil and a pen have a pocket clip. Is this likeness the reason why they both write? The lesson here is to beware of tempting generalizations that seem too easily to explain a great deal. In mathematics, and in the sciences and technologies, sound laws, principles, methods, and explanations must be specific and exact.

The abstract and the concrete, we noted earlier, must be played against each other for their mutual reinforcement. For example, $3 + 2 = 5$ is much more than a pure number fact to be memorized. It is a shorthand description of a pattern of combination which we are always coming upon in everyday life. The pupil must see through the abstract form $3 + 2 = 5$ to the many actual combinations that it stands for: two men joining three others, two pennies put with three pennies to make a nickel, and so on.

We may regard the relation $3 + 2 = 5$ as one which is abstracted from those natural situations in which groups or *sets* of objects are involved. But man is creator as well as abstractor. In his mind he can see groups where there are none. In measuring the length of a bench with a yardstick, he is, in effect, regarding the bench as separated into a chain of pieces, perhaps each a foot long, perhaps first three (one yard), then two of them. By this scheme he contrives to apply the discrete $3 + 2 = 5$ pattern to the continuous wooden bench top: 3 feet + 2 feet = 5 feet. Units are born in this way.

The subject of measurement and units is in need of clarification, from early grade to college years. Much nonsense has been written on the subject, some of which has become standard textbook material ("Only like quantities can be added"; "Two denominate quantities, like 2 feet and 3 feet, or like 2 feet and 3 pounds, cannot be multiplied together"; etc.). The state of our knowledge of the subject is still far from satisfactory, but enough is known today about the nature and conditions of mathematical application so that much could be done to clear up the treatment of units and applications at the elementary school level.

Mathematics Is a Form of Language. This is perhaps the most

important unifying idea of all, one that is associated with the modern point of view toward mathematics, and one that every teacher can usefully exploit. There are important differences in the ways we use word language and number language. But in early work the similarities need particular stress. A numeral or a numeral expression (like " $1 + 2$ ") names a number much as a proper noun or a definite descriptive phrase ("The Empire State Building") may name an object. This is a fundamental language idea which the teacher should continually dwell upon, and which we will take up later. Pupils should be guided early toward expressing their work in equation form. The addition facts, for example, should be presented both in vertical format and as equations, such as $5 + 7 = 12$. And these equations should also be called *sentences* or *statements*. This kind of approach makes it easier for the teacher to exact common high standards in the arrangement and presentation of both arithmetical work and English composition. J. A. Hickerson (3:15) cites the following possible steps of transition from a word statement to a statement in full arithmetical symbolism:

"Combining four blocks and three blocks make seven blocks."

"4 blocks and 3 blocks are 7 blocks."

"4 blocks + 3 blocks = 7 blocks."

" $4 + 3 = 7$."

So far our discussion has been orientational, illustrative in many ways of modern points of view, but without the full benefit of explicit introduction of modern mathematical concepts. We propose now to take up a few key concepts which have already begun to influence school arithmetic.

THREE S's SET • SCALE • SYMBOL

We choose the terms *set*, *scale*, and *symbol* as cue words to designate the three basic varieties of ideas, in large part modern, which form the principal theme of this article.

In the first year of life a child comes to recognize that some things belong together: spoon, glass, and serving dish; crib, blanket, bottle, and nap; several nesting blocks; daily family dinners.

These groups are examples of *sets*. Of such mental groupings some are only temporary, but others are so stable as to comprise permanent classifications or categories. (For material on categories and concept formation, see the various books of Jean Piaget; also (1) and (11).) That part of his experience which can be classified, and thereby related to other experience, makes a little island of order, sense, and meaning—the child's true world, for all the rest is formless chaos. It is not at first a terribly secure domain, most of the categories being a bit slippery or elusive, so that the child tends to cling to a few that loom especially large and important (for example, *Mother*). Then at last comes the magic of words, oral symbols that serve to tag or name the emerging categories, so that the child can get a firm mental grip on them.

Symbol then becomes the lever by which the child learns to move his world. He engages in that most characteristic and basic of all human activities, the use of language. Language is the instrument by which he is able to reach out to his fellows and they to him, by which he may keep hold of the past and change the future, by which he comes to understand and control not only his outward environment but his very self.

Symbol → Object

A symbol furnishes a name for a thing or object. "Theodore Q. Bixby," for example, may be the name of a particular person. The same person may have other names. To his wife he may be "Teddy"; to the federal government he may be known by his social security number or by his army serial number.

Our Hindu-Arabic numeral system furnishes a standard set of names for whole numbers. The Roman numeral system furnishes a different standard set. The numerals "23" and "XXIII" are two different names for the same number. Numeral expressions, like " $24 \div 3$," also name numbers. In arithmetic the equality sign is used to signify that two expressions both name the same number. The equation $1 + 2 = 3$, for example, is an arithmetical sentence which asserts that the numeral expression " $1 + 2$ " and the numeral "3" both name the same number.

Ordinary language and mathematical language share many important common features. The concept of levels of meaning or of

reference is basic to both. In ordinary language, in order to talk about objects (or actions or feelings) we use words (symbols) to denote or stand for or name these objects. When you say, "I see a chair," you are claiming to see an object, an actual chair, not the word "chair." The level of meaning with which you are concerned is thus what we will call the object level. But suppose you mention to someone that "chair" rhymes with "pair." Now you are talking about the word rather than the object. You are concerned with the level of symbol. This is the rarer case in ordinary language use. Quotes (or italics) are often used to draw attention to it. For example:

Object: Ohio is a midwest State. [The State itself]

Symbol: "Ohio" is spelled with two o's.

[The name of the State]

In arithmetic or algebra, the level of symbol or expression is that of numeral, and the object level is that of number.

It should be kept in mind that what you actually write down on paper or say aloud is not the symbol—not a word or numeral. The Hindu-Arabic numeral "3," for example, is a unique symbol. When you see "3" several times on a sheet of paper, you are only seeing different ink patterns. It is in your mind's eye that you see the same symbol as you scan each separate pattern. So what you actually see on the typographical level is transformed by the mechanism of perception to the level of symbol. This is a nearly automatic or unconscious process. Then the mind further spans the gap from the level of symbol to the level of object. Keeping these levels straight is a central problem, both in language and in mathematics. The subject of semantics deals with this, and with other difficulties having to do with the relations between symbols and the objects that they stand for.

Suppose that a boy is asked to take 5 away from 52, and that he gives 2 as the answer! Is this so unreasonable? It is likely that he has consciously or unconsciously interpreted the request to mean a numeral operation, the removal of a 5. He has to learn that take away, add, and similar expressions refer to operations on the level of number instead. Until he has grasped this basic orienta-

tional fact, it may do little good to keep reminding him that 52 is a shorthand for 5 tens and 2 ones, and so on.

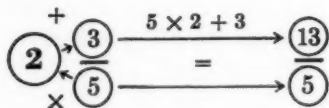
You can amuse yourself by making up puzzle examples illustrating the number-numeral or symbol-object distinction, like those shown below.

What do you see here? A MAD DOG
Which is larger? 273 or 5
$1 + 2 = \frac{1 + 5}{2}$ $\cancel{1} + 2 = \frac{\cancel{1} + 5}{2}$ $2 = \frac{5}{2}$

Some rules of arithmetic operate on one level, some on the other, while still others involve both. For example, can you always replace something by its equal? Take this statement: The denominator of the fraction $\frac{1}{12}$ exceeds 10. Can you here replace $\frac{1}{12}$ by its equal, $\frac{2}{24}$? You see that the reference in the statement to the fraction $\frac{1}{12}$ is on the level of numeral, not number, since it concerns the form of the expression. And on this level of symbol or expression, $\frac{1}{12}$ and $\frac{2}{24}$ are not equal. They are different forms which are not interchangeable.

A complex process, like long division, usually involves both levels. One test to see if the symbol level is involved is to ask whether the procedure can be carried out, unmodified, in terms of Roman numerals. If your answer is no, then it should be clear that the procedure must be at least in part dependent upon our Hindu-Arabic (decimal) notation, and hence belongs partly to the level of symbol. On the other hand, the observation that 9 is not divisible by 2 embodies a pure number property, independent of notation. You may equally well say that IX is not divisible by II, or that $5 + 4$ is not divisible by two.

It is especially important when dealing with fractions to minimize mechanical symbol manipulation, to think in terms of number and number operations. Consider the process of changing a mixed number to pure fractional form, say $2\frac{3}{5}$ to $\frac{13}{5}$. This is easily mechanized as follows:



However, the conversion is properly taught as simply as a case of number addition:

$$2\frac{3}{5} = 2 + \frac{3}{5} = \frac{2}{1} + \frac{3}{5} = \frac{10}{5} + \frac{3}{5} = \frac{13}{5}.$$

The standard computational processes, like column addition or long division, must eventually become highly mechanized so that the advanced pupil can calculate speedily with a minimum of effort. But of course they must not initially be taught that way. Not only would the pupil instructed by rule and rote become frustrated by his apparently senseless symbol maneuverings, but he would have an insecure foundation for continuing his work, a foundation that would crumble and be lost upon a minor memory lapse. In the early stages of learning, the level of number should receive as much emphasis as possible, and along with this, concrete interpretations should be widely drawn upon.

The long division process, for example, may be regarded as a scheme of repeated subtraction. To divide 978 by 42, we wish to subtract 42 from 978, then subtract 42 again from the result, then again, and so on, counting the number of times this can be done. Of course, it is tedious to take 42's away one by one. We soon learn to take them away ten at a time, or a hundred at a time, or whatever is possible. Here, since 4200 exceeds 978, we first take 420's away as long as we can, then 42's. The procedure is shown at the left below. It is easy to see how this process evolves by way of the shortened form shown in the middle, to our standard procedure displayed at the far right.

	<i>Take Away</i>	<i>Partial Remainder</i>			
		978		23	
			$42 \overline{)978}$		$23 \overline{)978}$
20	420	558	-840 =	20 × 42	84
	420	138	138		138
			-126 =	3 × 42	126
			12		12
3	42	96			
	42	54			
	42	12			

QUOTIENT: 23

Remainder: 12

The division process shows us that 978 can be decomposed into 23 forty-twos, except that 12 of the original 978 is left over. In an equation:

$$978 = 23 \times 42 + 12$$

$$\text{Dividend} = \text{Quotient} \times \text{Divisor} + \text{Remainder}$$

If 978 students, for example, are to be sectioned into classes of 42 students each, there will be 23 full classes, also one incomplete class with only 12 students. Note that each unit of the quotient 23 refers to 42 students, while each unit of the remainder refers to but one student. Quotient and remainder are very different in meaning and in relative importance. Furthermore, in division with decimals, the work is always carried far enough so that the remainder may be discarded. So should we not frown upon the practice seen in many school texts, of writing the remainder up along with the quotient? What could be more confusing to the pupil?

$$\begin{array}{r}
 23 \text{ R } 12 \\
 42 \overline{)978} \\
 \underline{84} \\
 138 \\
 \underline{126} \\
 12
 \end{array}$$

A Poor Practice!

What is a Set?

Because the concepts of set and symbol underlie and precede language, neither can be satisfactorily defined in words. They are true primitive concepts in that they are part and parcel of everyone's earliest experiences.

A set is a collection or a class. But these are only synonyms. We cannot properly answer the question, What is a set? But we can tell how the term *set* is used.

Whenever several things are linked together in the mind, then we may speak of them as the elements of a set.

To describe a set we may list its elements. Or we may give some rule or test for determining which objects are its elements. The rule is often quite casual. In conversation, we rely upon each person's divining the other's intent. Examples: "Look at *those people over there*." "Do you ever see *your friends from Scranton*?" "Got *your golf clubs* with you?"

Referring to a deck of playing cards, someone may speak of the aces. In this way he describes a set whose elements are the Ace of spades, the Ace of hearts, the Ace of diamonds, and the Ace of clubs. Conversely, the set whose elements are the King of spades and the King of clubs can be described as the set of the black Kings. The phrase *the even numbers* describes a perfectly definite set, yet one whose elements, 2, 4, 6, ... cannot be listed in full.

Sets and Numbers

A child places a hand over each of his eyes, and discovers that he has as many eyes as hands. This matching of two sets is the basic notion from which emerges the idea of number.

As the child continues to find sets that match the set of his hands, such as his feet, his ears, his mother's hands, his shoes, or a pair of blocks, he develops a need to name the common property of all these sets. He soon learns that the customary name is *two*.

Whenever the elements of two sets can be associated in pairs, each element of one set being associated with one definite element of the other, and vice versa, then we say that the two sets match, or are in one-to-one correspondence. Given a set of boys and a set

of girls, if they pair off and dance in boy-girl couples with no one left out, then we can be sure that the set of boys matches the set of girls. If every seat in a classroom is taken and no student is standing, we observe that the set of chairs matches the set of students.

Color, mass, shape, and so on, are familiar properties of objects. To find out if you and a friend have the same conception of red, you may show your friend many different objects. You hold up an object that you call *red*, and he also calls it *red*. You show another that you call *not red*, and he also calls it *not red*. If you and he always agree in this way, you may be fairly sure that red has the same meaning for you both. (Of course, there is always the risk of overlooking some region of disagreement. In science, the discovery of such a region often leads to a reformulation of the conception under test.)

Number is a set property. In contrast to physical and physiological conceptions, like color, which are necessarily approximate, the abstract conception of number is perfectly defined. The test to tell if two sets are alike in their number is objective and exact: the sets either match or they do not match. With every given set we associate a number. Given any other set, the rule of matching determines whether or not the same number is assigned to it. By common agreement, for example, we name the number that goes with a full set of fingers *ten*. The rule requires us to associate the same number with every matching set, so we say also that we have ten toes, and that ten pennies make a dime.

(Many writers define a number to be the class of all sets that match a given set. This is objectionable on at least three counts. It is too technical and mathematically sophisticated to be really understood by even one out of ten thousand teachers. Secondly, because it has the form "a number *is* such and such," many who hear it erroneously conclude that it is the *only* correct definition. Finally, it is logically unsound because it leads to the *set of all sets* paradox. The testing scheme mentioned here is sound, and is easily understood. It is also the kind of scheme now used in all the sciences in formulating basic conceptions.)

Set relations and operations can now be made the basis for

number relations and operations. Figure 1 displays the kind of set relationship which is the basis of the number comparisons *less than* and *greater than*.

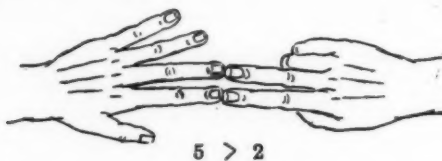


Figure 1

One of the simplest and most important of set operations is that of forming the union of two sets. The elements of two or more sets are lumped together to form a single new *union set*. Two piles of blocks are swept together to make one pile. Two families climb into one car for a drive. Husband and wife combine their Christmas card lists into one. Clearly this union operation leads to the idea of number addition. If two sets do not overlap (share an element), the number of the union set is the sum of the numbers of the two sets.

In developing number notions, most teachers will instinctively call upon some very simple set conceptions. A teacher will demonstrate $2 + 3 = 5$, for example, by pushing together a pile of two blocks and a pile of three blocks to get a single pile of five blocks. But this is only the beginning. The teacher with a background of elementary set theory can use it to illumine every area of arithmetic, both for himself and for his pupils. In Chapter 11, Boyer, Brumfiel, and Higgins discuss basic set conceptions, while in Chapter 3, Van Engen and Gibb show how such ideas can be brought into the classroom to reveal meaningful structure and pattern. So we will not here enter into a systematic development, but will take up some illustrations and mention a few matters not covered in other discussions.

Counting

When we think of a number as telling the size of a group or set, then we call it a *cardinal* number. When we think of a number as

designating a position in a line or row, then we call it an *ordinal* number.

That building has 12 floors. In this statement, the numeral "12" refers to a cardinal number. *My office is on the 12th floor.* The numeral "12" now refers to an ordinal number. The appearance of the suffix *th* is a sure sign that the numeral to which it is attached refers to an ordinal. But the *th* does not always appear. Floor No. 12 and Floor 12 both represent ordinal usages. But do not try to classify every whole number usage as cardinal or ordinal. A phone number, for example, is a mere code symbol.

Like the cardinal, the ordinal conception is primitive. No peoples are known to have developed the one approach to number without the other. It would seem, in fact, that the ordinal idea of *one more than*, of one thing following another, is physiologically based, or built right in us. It is involved in the sense of time, the rhythm of the heart beat. Rhythm can, indeed, be exploited as a basis for number learning (5). Tots often learn to count in rote fashion—*one, two, three, . . .*—before they are able to relate these number words to sets. Do some children first connect *three* with a position in a sequence, and others first connect *three* with a group of objects? Perhaps this question is not important. For whatever the sequence of learning may actually be, nearly all children learn very early to make both connections, and to mix them freely. Relations like $2 + 3 = 5$ can be given either cardinal or ordinal interpretation. Mixed interpretations are common: "From the 2nd step [ordinal], take 3 steps [cardinal] up, and you will reach the 5th step [ordinal]."

The connection between cardinals and ordinals constitutes the basic counting principle, illustrated in Figure 2. This illustration shows a row of blocks, numbered 1, 2, 3, 4, 5. We may think of the numeral on a block as referring to an *ordinal* number describing the position of the block in the row. But we may also think of this numeral as referring to a *cardinal*: the number of the set of all the blocks up to and including the one under consideration.

Now suppose you are to count the rolls on a plate. You point to a roll, and say "1," to another and say "2," to another and say "3," to another and say "4." Suppose you stop here, having run

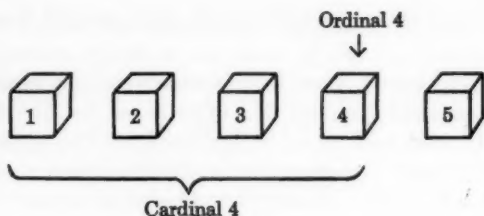


Figure 2

through the whole group of rolls. What you have done is match the set of rolls with the set of numbers 1, 2, 3, 4—or, for that matter, with the set of blocks numbered 1, 2, 3, 4. You know, therefore, that the cardinal number of the set of rolls must equal the cardinal number of the set of blocks. But the counting principle asserts that this cardinal has the same name as the ordinal of the last block, “4.”

Evidently our familiar way of counting is a correct scheme for finding the number of a set. In fact, it is ordinarily by simple counting that young children are guided to discover basic addition and subtraction combinations, like $2 + 3 = 5$ or $7 - 3 = 4$. We may look upon counting as the basis from which more general scale conceptions will later develop.

Using Set Combinations

The Arithmetic Teacher for November 1958, contains a cartoon reprinted from the *Chicago Tribune*, showing two little boys leaving a schoolyard. One is saying to the other, “She told you 5 and 2 make 7? THAT does it! She told me 4 and 3.” Obviously the little boys’ teacher was unacquainted with the set operation of partition. This is converse to union, mentioned earlier. Union lumps several sets into one. Partition splits a set into several. If seven blocks are lined up, you may show a partition into two sets by placing your hand between any two blocks. Thus Figure 3 suggests a partition corresponding to $5 + 2 = 7$. Here stars represent the blocks and a vertical line represents the hand.



Figure 3

The cross-product (or Cartesian product) of two sets is a much more complicated set combination than union. It is formed by pairing every element of one set with every element of the other. The cross-product of a set of boys and a set of girls is the set of all possible boy-girl couples. As set union gives a basis for number addition, the cross-product gives a basis for number multiplication.

The cross-product idea leads to an interpretation of multiplication which every teacher needs to know and use: *A product of two numbers can be associated with a rectangular pattern.* Figure 4

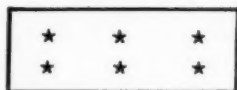


Figure 4

shows a star diagram representing the product 2×3 . We need not invoke the technical cross-product conception to see this. We need but observe that there are two rows of three stars each. Of course we also see three columns of two stars each. This means that the set of stars in the figure represents 2×3 and 3×2 . Hence $2 \times 3 = 3 \times 2$. Is it hard to pass beyond this special numerical case to the general commutative law of multiplication, $ab = ba$?

The distributive principle (or law) was mentioned earlier. This tells how to multiply a sum by a number: multiply each term of the sum by the number, then add the results. Example:

$$2 \times (3 + 4) = 2 \times 3 + 2 \times 4.$$

The reason for the rule is apparent from a star diagram like Figure 5, which illustrates the example just cited.

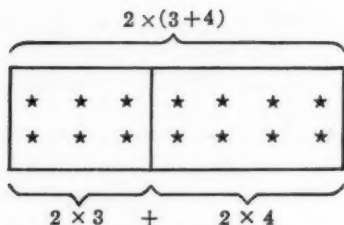


Figure 5

The distributive law is applied in all our common computational processes. To find 2×34 , we double both the 3 and the 4, getting 68, because:

$$\begin{aligned}
 2 \times 34 &= 2 \times (3 \times 10 + 4 \times 1) \\
 &= 2 \times (3 \times 10) + 2 \times (4 \times 1) && \text{[Distributive Law]} \\
 &= (2 \times 3) \times 10 + (2 \times 4) \times 1 && \text{[Associative Law]} \\
 &= 6 \times 10 + 8 \times 1 \\
 &= 68.
 \end{aligned}$$

In mentally adding 200 and 300, we use the law in the reverse direction, or as a factoring rule:

$$\begin{aligned}
 200 + 300 &= 100 \times 2 + 100 \times 3 \\
 &= 100 \times (2 + 3) && \text{[Distributive Law]} \\
 &= 100 \times 5 \\
 &= 500.
 \end{aligned}$$

Of course it would scarcely be desirable to explain the above procedures to young children as applications of abstract laws. But the patterns of correct explanation are set in terms of such laws. It is up to the teacher (with an assist from the textbook writer) to devise concrete or familiar schemes by which to convey such explanatory patterns to pupils at their own level of comprehension.

In terms of sets, division may be regarded as associated with a partition of a set either (a) into mutually matching subsets, called cells, or (b) into subsets of a given size.

Suppose you wish to distribute 12 cookies equally among 3 children. This problem can be related to the 3×4 star diagram

in (a) of Fig. 6. Here the quotient of 12 by 3 is given by the number of stars in each cell. But suppose that you wish to give

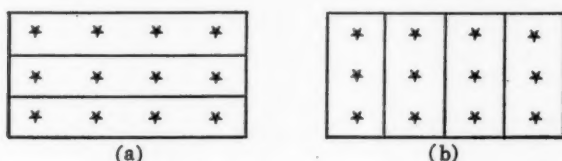


Figure 6

three cookies to each of several children and want to know how many children can be so rewarded from your stock of 12 cookies. The same 3×4 star diagram applies, but with the different partitioning setup shown in (b) of Figure 6. Now the quotient of 12 by 3 is given by the number of cells. It is, of course, the point of view in (b) that is the basis of our common long division process, which we have already characterized as a scheme of repeated subtraction of the divisor.

The Number Scale

We mentioned earlier that addition and subtraction facts, like $2 + 3 = 5$, can be discovered or verified by counting along a row of numbered blocks. At an appropriate grade level, a number line or scale can be substituted for the blocks (Figure 7).

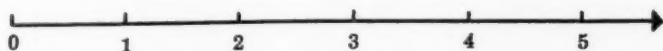


Figure 7

The subtraction form $5 - 2$, for example, may be interpreted as directing the pupil to count 5 steps or units forward, then 2 backward, ending with the result 3. (Mueller, in (4), develops all four basic number operations in terms of scale operations.)

When a child first sees numbers put on a scale, he stands at the threshold of a completely new conception of number. He will eventually learn that every point of a line has a number associated with it, and that these are called real numbers. This tie-in between

the number system of arithmetic and algebra and the continuous line of geometry is a major mathematical achievement, basic to almost all analytical and applied mathematics.

The scale number idea can, of course, be correlated with earlier cardinal and ordinal notions (as well as with fractional and decimal ideas as they have been or are being developed). The statement $2 + 3 = 5$, for example, can now be taken as referring to a measurement situation, in which a length of 2 inches is joined with a length of 3 inches to produce a length of 5 inches. In this interpretation, the numerals "2" and "3" now designate scale (or real) numbers instead of cardinals or ordinals. A useful project can be the making of a slide rule for adding and subtracting, with scales graduated, say, to sixteenths for handling mixed numbers and/or to tenths for handling decimals.

The one-way scale is mathematically inadequate. A common thermometer scale is two way, because it extends in both directions from zero. Even the letter-space scale on some typewriters is two way, with the zero mark, or origin, at the middle. We ought to recognize the inadequacy of the one-way scale a little earlier than we now do in the classroom. In the fifth or sixth grade we could give children a two-way ruler to use (Figure 8). Mark the scale to the left with red numerals (later termed negative) and the scale to the right with black numerals (later termed positive).

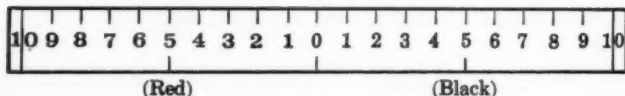


Figure 8

Children could use such a ruler for ordinary measurement purposes, often placing one of the red or black figures, rather than just zero, at the initial end of the length to be measured. They could use it as an aid in scoring games having *put and take* or *profit and loss* features. In this way they would become painlessly accustomed to the addition and subtraction properties of signed numbers.

The notion of introducing negative number ideas in the mid-

dle grades is far from new. But earlier unsuccessful experiments lacked the benefit of the structural approaches now available. A set approach to signed number arithmetic is also feasible, making use of red and black checkers. A set of eight black checkers and three red checkers would have a *signed number* of $+5$ (the number of black checkers left after three have been paired off with reds and thus nullified).

Fractions

How may we use the set•scale•symbol trilogy in developing the subject of fractions (10)?

From the point of view of symbol, we look upon a fraction as a numeral form, and we focus upon the language aspects of fractional use. Of course we tie this in from the beginning with set ideas, looking also upon a fraction as a number. (In speaking of a fraction as a number, we often refer to the *value* of the fraction.)

Suppose that John selects 8 marbles from a group of 12. You say, "John took $\frac{8}{12}$ of the 12 marbles." Now if the original dozen had been in 6 groups of two each, and if John had taken 4 of these groups, you could have said, "John took $\frac{4}{6}$ of the 12 marbles." (See Figure 9.)

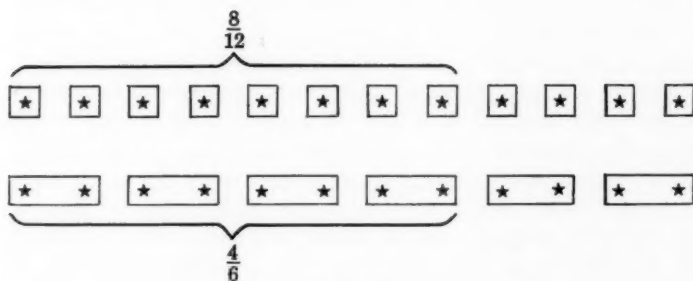


Figure 9

Now consider this sentence: "John took of the 12 marbles." Whether you write $\frac{8}{12}$, $\frac{4}{6}$, or $\frac{2}{3}$ in the box, the same information will be conveyed by the sentence, namely that John took 8 mar-

bles. This, then, is a basis for defining equality of fractions. We say that $\frac{2}{3}$ and $\frac{4}{6}$ have the same value, and we write $\frac{2}{3} = \frac{4}{6}$.

In developing basic fractional ideas and in deriving ways of operating with fractions, the teacher can do a great deal with no more than just a set of twelve blocks. Consider: $\frac{2}{3} \times \frac{3}{4}$. What is $\frac{2}{3}$ of $\frac{3}{4}$ of 12, that is, $\frac{2}{3}$ of ($\frac{3}{4}$ of 12)? The teacher may divide 12 by 3, then multiply the result by 2 to get 8, and at the same time take $\frac{3}{4}$, or 8, of the 12 blocks. Then she may divide 8 by 4, multiply the result by 3 to get 6, and at the same time take $\frac{2}{3}$, or 6, of the 8 blocks. It should become apparent quickly that it is necessary to divide by each denominator in turn and to multiply by each numerator in turn. But this is equivalent to dividing by the product of the denominators and multiplying by the product of the numerators. So the rule for multiplying two fractions has essentially been developed. The twelve blocks allow illustration of many addition and subtraction exercises involving halves, thirds, and fourths.

In early work with fractions such set and structural ideas are vitally important. But it is equally vital that the student should eventually mature beyond this stage, and learn to look upon fractions simply as scale numbers, whose form of expression ($\frac{2}{3}$ or $\frac{4}{6}$, for example) is incidental. Structurally, $2 \times \frac{1}{2}$ and $\frac{1}{2} \times 2$ may be regarded as having different meanings. But computationally, we must look upon $2 \times \frac{1}{2} = ?$ and $\frac{1}{2} \times 2 = ?$ as posing essentially the same problem. Only on the scale are size comparisons, as between $\frac{1}{2}$ and $\frac{1}{3}$, fully appreciated, and the connection between fractions and decimals satisfactorily grasped. Nor can a student manipulate fractions freely and efficiently until he looks on them simply as numbers without special properties as such. He need not have to learn nearly two dozen particular rules for dealing with different combinations of whole numbers, mixed numbers and common fractions as are found in many arithmetic texts.

THE CURRICULUM AND THE FUTURE

What will tomorrow bring? Probably some speed-up, possibly differentiated or multiple-track programs under which there will be a small gain for the average student, both in time and in the

amount of mathematical material covered, and a very substantial gain for the superior student. More geometric material may be brought in at many grade levels (8). Otherwise the most significant changes in grades 1 through 6 may likely comprise not new topics but new approaches, with a continually increasing emphasis upon set, scale, and symbol ideas, with much attention paid to the psychology of development of the conceptual material. We may look for more aid from the research psychologists than they have been able or willing to furnish in the past.

In junior high school grades 7 and 8, not only the approach but also the topical content is due for radical revision. We find a strong trend toward the elimination of many social and business arithmetic topics. The area is wide open. It is fascinating to consider the possibilities. Probably increased attention will be given to the field of practical computation, such as the use of the slide rule and of office and electronic computers, the application of approximation techniques, and the treatment of error. We may hope for improved ways of teaching mathematical application and for clarification of the somewhat muddled subject of measurement and units. Insofar as *pure* mathematical topics are concerned, we may anticipate greater emphasis upon the number scale (real number system), including the use of signed numbers. Informal and mensurational geometry may receive more attention. And we may expect that a more extensive groundwork will be laid for future algebraic studies, with special attention given to the language aspects of the subject. Among additional topics that may be introduced are combinatorial problems (numbers of ways of doing things, numbers of arrangements and combinations, for example), simple probability and risk, diophantine problems, modular or clock arithmetic, and numeral systems based on 2, 4, 5, 6, 8, or 12, instead of on 10. Finally, there is much material from elementary number theory that is adaptable for the seventh and eighth grades. It may even be possible to take up some simple proofs about properties of odd and even numbers (for example, *the sum of two odd numbers is an even number*). It is important that the idea of proof should be introduced as early as possible in school studies, and that it should be associated in the child's mind with mathematics (and science) in general, not with geometry alone. It may be

anticipated that about half the material cited as suitable for the ninth grade in the recently published Report of the Commission on Mathematics of the College Entrance Examination Board may within a few years be included in the work of the seventh and eighth grades.

Looking ahead, we may see a drawing-together of mathematics and language instruction in the pre-school and early grade period. Even today the refrain that *mathematics is a language* is heard from all quarters. In the early grades the similarities between the ways we use the language of number and the language of words are more important than the differences. Possibly this language theme will come to dominate elementary mathematics instruction.

As young pupils learn to look upon mathematics as a special kind of language, one in terms of which concepts may be given precise formulation, thinking sharpened, and communication perfected, we may well hope that many or most of them will come to regard the subject more as a beautifully designed functional tool than as a Rube Goldberg contrivance whose operation is an unfathomable mystery.

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Background Mathematics for Elementary Teachers

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LACK OF AGREEMENT as to what constitutes the most effective training for prospective teachers of arithmetic has resulted in a great diversity of programs, and considerable evidence is available for decrying inadequate instruction in this important phase of elementary-teacher training. This chapter deals with the professional preparation of teachers of elementary school arithmetic. Pertinent literature and research are reviewed in the first part of the chapter. There follows a summary of current practices and requirements in the preparation of elementary teachers. The content of a proposed background course in mathematics for prospective teachers was itemized in the form of a questionnaire; the results are summarized next. The authors' recommendations follow, and the chapter closes with a bibliography of selected references.

SURVEY OF RESEARCH AND LITERATURE

The careful preparation of prospective teachers in mathematics subject matter is a prerequisite to an improved program in arithmetic in elementary schools. This point of view has been presented consistently in the writings of research workers in the field of arithmetic for the past two decades. Brownell (7) pointed out in 1945 that the teacher who knew arithmetic as it was taught in the first three decades of this century would be in no position to expound the subject in its newer, meaningful aspects. Committees of national importance have recommended that training in background mathematics be provided in the preparation of students to teach in the elementary schools. The President's Scientific Research Board (22:99) in 1947 pointed out that

Since one of the major handicaps to an effective science and mathematics program in the elementary school is the inadequate preparation of teachers, certain specific steps to improve this preparation are necessary.

The Commission on Post-War Plans of the National Council of Teachers of Mathematics (18) presented, in 1947, the same point of view—that teachers need background mathematics and that the main requirement is an understanding of arithmetic. In 1948 the National Commission on Teacher Education and Professional Standards (34:150) proposed that a course in general mathematics at the college level be required of all teachers.

Low Requirements

In view of the need and demand for sound background mathematics for prospective elementary school teachers, one might expect to find many institutions of higher education providing such courses. This is not the case. Research studies show alarming inadequacies in background courses in arithmetic for prospective teachers. Grossnickle (12) made a questionnaire study of 129 state teachers colleges. The results were checked against specific catalogues of responding institutions. This investigation revealed that three-fourths of the teachers colleges required no credits in secondary-school mathematics for admission. The curriculum in

one-third of the state teachers colleges offering work for kindergarten and primary grades required an average of 1.4 semester hours of background mathematics. Two-thirds of the colleges offering a program for general elementary grades required an average of 2.0 semester hours of background mathematics. The average total requirement in mathematics to graduate from a state teachers college is approximately three semester hours. About 33 percent of the students in the curriculum for kindergarten and primary grades and 10 percent in the curriculum for general elementary grades could graduate without taking any courses in mathematics.

Layton (14) reported that only one-fourth of 85 institutions of higher education preparing elementary school teachers required mathematics for admission to the teacher-education curriculum. The mean mathematics content requirement of the four-year elementary curriculum was approximately 1.6 semester hours, as compared with means of 4.3 semester hours for art and geography, and 11.5 semester hours for English.

Stipanowich (27) used a jury of 70 mathematics-education specialists affiliated with 66 institutions of higher education in 32 states to determine which of several practices currently employed in college training programs in mathematics for prospective elementary school teachers were favored by them. He concludes that the study provides additional evidence to support the claim that many teacher-training institutions are not providing adequate preparation in mathematics subject matter for students preparing to teach in the elementary school.

Need for Mathematical Understandings

Glennon (11) has reported pertinent research on basic mathematical understandings in teacher education programs. He concludes that there is no significant difference in the achievement of basic mathematical understanding between:

1. Teachers college freshmen and teachers college seniors
2. Teachers college seniors who have taken a course in the Psychology and Teaching of Arithmetic and those who have not taken such a course

3. Teachers-in-service who have done graduate work in the Psychology and Teaching of Arithmetic and those who have not done such work.

Also, near zero correlation was found between achievement in mathematical understanding by teachers-in-service and the length of teaching experience.

Glennon concludes that present pre-service and in-service training programs are inadequate from the standpoint of attention to the development of basic mathematical understandings, and that adequate training programs must include background mathematics as well as required work in the specific problems of teaching arithmetic.

Since Glennon's study was published in 1949 and was based upon data collected in 1948, these findings may not be indicative of present pre-service and in-service training programs. Similar studies should be made to measure the effectiveness of current programs.

Schaaf (25) presented evidence that elementary school teachers did not understand mathematics, and made recommendations for a course in arithmetic for teacher education institutions. Weaver (29) also presented evidence of teachers' lack of understandings vital to meaningful arithmetic instruction in the elementary school. He identified the responsibility of teacher education institutions to improve arithmetic scholarship.

Brown (6) of Illinois Normal University has reported on methods used to prepare able and confident teachers of mathematics for elementary school classrooms. Two basic concepts are stressed in courses designed to help prospective teachers: first, students must understand the meaning and use of fundamentals of elementary mathematics; and second, they should like mathematics. Two levels of instruction provide for individual differences of students enrolled in the course: one for those who are advanced in ability and background, and another for those with inadequate background. Tests of arithmetic computation and reasoning are used to select students for the two groupings. Oral questioning pertaining to likes, dislikes, and the amount of mathematics completed is used to supplement tests results.

Poor Teaching Conditions

Not all the difficulty in mathematics teaching in elementary schools "can be eliminated by retraining teachers, even if a good rereading process were known." (10:17) One observer for the Educational Testing Service of New Jersey (10) visited 60 classrooms to verify at first hand what the books and experts were saying about the deplorable state of mathematics teaching. He found 10 classrooms in which teaching was reasonably effective; in the other 50 the instruction was so confused that learning of any kind seemed to be largely accidental and unilluminated by any learning theory. The working conditions of teachers were extremely poor. Most teachers were struggling with classes of 35 to 40 pupils; some of these classes were a combination of two different grade levels and almost always ranged from the bright but bored to the dull and bewildered. A fundamental change in teaching conditions is crucial. Only the public, acting through its representatives on school boards, can bring about this change.

Attitudes of Teachers

Prospective elementary school teachers have many unfavorable attitudes toward arithmetic. Causes for these attitudes seem to be associated with several factors: lack of understanding of arithmetic processes; little application to life and social usage; poor teaching techniques involving boring drill; too many verbal problems; little or no teacher assistance; feelings of inferiority and insecurity; and failure to keep up with others in the class.

These data collected by Dutton (8) indicate deep-seated, highly emotionalized attitudes that have persisted from childhood and are prominent in the thinking of prospective teachers as they progress through the teacher education program. Additional courses in mathematics and the lapse of several years since arithmetic was last studied have not erased the distasteful feelings toward arithmetic.

Attitudes of prospective teachers toward arithmetic were investigated by Dutton (9) in another study. An attitude scale was constructed to measure the amount of *like* or *dislike* of arithmetic of students enrolled in education classes. The findings show that

attitudes toward arithmetic can be measured objectively and that useful data may be obtained which will be helpful in the education of prospective elementary school teachers. (See Chapter 8, page 181.)

CURRENT PRACTICES IN TEACHER TRAINING

Certification Requirements

During World War II it was general practice in most states to issue an emergency certificate to almost anyone who would agree to teach. Since the end of World War II, however, there has been a steady increase in the amount of college training required to qualify for a regular elementary school teaching certificate. Four years of training is now the prevailing type of program for the training of elementary school teachers. There is some indication of a trend toward five-year programs, but no state prescribes that amount at this time.

Table 1 shows the change in minimum requirements for certification of elementary teachers between 1951 and 1957. The overwhelming shift to a minimum of four years of training is most noticeable. Of the two states that listed no requirements in 1951, one now requires four years of training and the other, one-half

TABLE 1

Minimum Requirements in Years of Training for Certification of Elementary Teachers in the United States

Years of College Training	Number of States	
	1951	1957
None.....	2	0
One-half.....	0	1
One.....	7	2
One and one-half.....	2	0
Two.....	16	9
Two and one-half.....	1	0
Three.....	3	1
Four.....	17	35

SOURCE: Adapted from Armstrong and Stinnett (2 and 3).

year. Ten of the 13 states requiring less than four years of college preparation are in the Great Plains Region of the United States.

An analysis of prescribed standards for certification by the various states does not reveal the actual level of training of each teacher in each state. Because of the increasing birth rate each year since 1947 it has been impossible to secure enough fully trained teachers for the elementary grades. Grossnickle (12) pointed out that about 30 percent of all elementary teachers in 1948-49 held provisional certificates. About the same proportion of elementary teachers held provisional certificates in 1957. It is apparent that with the rapidly increasing standards for certification, more and more people are entering the profession with more training than before. Also, the teachers-in-service are continuing to raise their level of training.

Some states have legislated specific subject matter requirements for elementary school certification. Grossnickle reviewed this legislation in 1951. An examination of state requirements for the training of elementary teachers reveals that this type of legislation has not changed in the past 10 years. In only 12 states is a specific requirement in the field of mathematics mentioned. Seven states require from three to six semester hours of general mathematics at the college level, while the other five states prescribe a methods course in arithmetic for teachers. Four other states prescribe a "general education" background in which mathematics may be one of several general areas selected to fulfill this requirement.

Nature of Training in Mathematics

The mathematics background of elementary teachers-in-training and elementary teachers-in-service stems from several different sources. A small group of elementary teachers have had no mathematics beyond the eighth grade; a great majority have had one to three years of mathematics in high school, but none in college; and another small group have had the normal eight years of arithmetic plus some high school mathematics and a small amount of college mathematics. It can be safely generalized that no teacher has less than an eighth grade level of preparation and very few have as much as a minor in mathematics at the college level.

The writers analyzed the 1957-58 catalogues of 96 teacher-training institutions (two from each state) selected at random from a library of more than 400 catalogues. Sixty percent of the institutions were state colleges and 40 percent were private or state universities. The catalogues were analyzed for three factors:

1. The amount of high school mathematics required for admission to the elementary teacher training program of the institution
2. The type and amount of mathematics required at the college level for elementary teachers-in-training
3. The type and number of arithmetic methods courses required of elementary teachers-in-training.

College Entrance Requirements. No state prescribes a mathematics requirement for graduation from high school or admission to college. In this study it was found that $33\frac{1}{2}$ percent of the institutions require either one or two years of high school mathematics for admission to the elementary teacher training program. An additional $16\frac{3}{4}$ percent of the institutions require proficiency in the field of mathematics, determined by a fundamentals test, for admission to the college. If the prospective student does not pass the mathematics fundamentals test, he must successfully complete a course in the fundamentals of mathematics for admission. A total of 50 percent of the institutions requires a course in high school mathematics or proficiency in mathematics for admission.

One might conclude from these data that prospective teachers in 50 percent of the teacher-training institutions have inadequate background in the field of mathematics, but this is not the case. A study was conducted in two teacher-training institutions, one a state college, the other a state university, in two different sections of the United States, to determine the mathematics background of entering students where no admission requirement in mathematics exists. It was found that from a total of 1186 juniors in the two institutions, 86 percent had completed two or more years of high school mathematics. Only $1\frac{1}{2}$ percent of these students had entered the college with no high school mathematics. Four percent had completed four full years of mathematics in high school.

In college, however, only 11 percent had taken work in mathematics, and one-half of this group completed only one semester. Less than one-half of one percent of the 1186 juniors had completed a minor in college mathematics. Admittedly, the entrance requirements may seem low, but it appears that the vast majority of elementary teacher training candidates have had two or more years of high school mathematics. On the other hand, the possibility that these candidates will take mathematics in college is very slight. It must not be assumed, however, that two or three years of high school mathematics is adequate preparation in this field for people entering elementary school teaching.

College Mathematics Requirements. In the group of 96 colleges and universities 56 percent require a course in college mathematics regardless of previous knowledge or background; an additional 5 percent require a five-semester-hour course in which mathematics content and methods are both taught; and 39 percent do not require any type of mathematics background at the college level. It is encouraging to note that 22 percent of these colleges and universities require minimum proficiency for admission as well as a course in basic mathematics.

Typically, the required college mathematics course is a two- or three-semester-hour course called *General Mathematics* or *Mathematics for General Education*. The primary purpose of the course is to acquaint the student with uses of mathematics in everyday life and in general reading. Some emphasis is given to fundamental operations with integers and fractions. An appreciation of the historical development of various number systems is usually stressed. It is important to note that the majority of course descriptions state that the development of mathematical skills is important but that the course emphasis is on the understanding of mathematical concepts and procedures.

College Arithmetic Methods Courses. Among the same group of colleges and universities, 56 percent require a specific methods course in the teaching of arithmetic; 5 percent offer methods in a combined course with the mathematics background requirement; another 10 percent have no arithmetic methods course as such, but offer an all-inclusive course which covers methods of

teaching all elementary school subjects. The remaining 29 percent have no arithmetic methods course requirement.

Fifty percent of the schools with an arithmetic methods requirement offer a three-semester-hour course; 40 percent of the institutions have the course organized on a two-semester-hour basis; while the remaining 10 percent offer a five-semester-hour course. It is reasonably safe to assume that in those institutions where instruction in the teaching of arithmetic for elementary grades is a part of a general methods course, the amount of time given to arithmetic is equivalent to less than one semester hour of credit.

Summary

An investigation of state certification requirements and practices of teacher training institutions reveals the following pertinent facts:

1. Thirty-five states require a bachelor's degree for certification of elementary school teachers. This is more than double the number of states with such a requirement six years previously.
2. Although the length of the college training program for elementary school teaching has increased greatly in recent years, the change in prescribed mathematics has been negligible.
3. Twelve states have legislated a specific requirement in mathematics as a part of elementary teacher training. Seven of the states require work in general mathematics, while five list a requirement in methods of teaching arithmetic.
4. A prospective elementary teacher may be admitted with no credits in high school mathematics to two-thirds of the teacher-training institutions studied by the authors. However, one-fourth of these institutions require for admission a minimum level of proficiency in arithmetic.
5. In institutions that have no admission requirements in mathematics, 86 percent of the candidates have completed at least two years of high school mathematics.
6. Sixty percent of the institutions require for graduation a three-semester-hour course, general mathematics in nature.
7. Fifty-six percent of the colleges in this study require a course in the teaching of arithmetic.
8. Twenty-two percent of the colleges require a background

course in mathematics and also an arithmetic methods course. Five percent of the institutions do not require either type of course.

9. The typical student completing a four year program for certification to teach in the elementary grades has completed two years of high school mathematics and has had either a three-semester-hour course in general mathematics or a two-semester-hour course in the teaching of arithmetic.

A PROPOSED MATHEMATICS BACKGROUND COURSE

The writers prepared a questionnaire which was submitted to the elementary teacher training department of all members of the American Association of Colleges of Teacher Education. The purpose of the questionnaire was to establish a solid base of professional opinion for recommending a background course in mathematics for all elementary school teachers who are to be certified to teach in grades one through eight.

Following a recommendation in *The Teaching of Arithmetic*, the Fiftieth Yearbook of the National Society for the Study of Education, the questionnaire was organized on the assumption that a six-semester-hour course or sequence would be necessary.

In a sense the findings of the questionnaire may serve as the basis for a background course in mathematics. It was pointed out by many of the respondents that a list of topics does not make a course. Since many students would have encountered a majority of these topics in high school and elementary school, the course would be dull and meaningless unless these topics were presented in a new light with emphasis upon meaning and understanding. It is hoped that the following tabulations with accompanying comments will provide an impetus for the evaluation of current courses or the planning of new courses for training teachers of arithmetic.

It was suggested in the questionnaire that the six-semester-hour course under discussion should have two major objectives: (a) to provide prospective teachers of arithmetic with an adequate background in mathematics; and (b) to give prospective teachers

an appreciation of the cultural and aesthetic value of mathematics as well as its more practical values.

The questionnaire was organized as a list of appropriate topics to be included in a background mathematics course for prospective elementary teachers. The items were grouped under several major topic headings. Respondents were requested to react to each item on the questionnaire in terms of its desirability as a part of a mathematics background course for teachers. Of the 313 members of the AACTE, 227 responded. Tables 2 through 16 present the items as they were listed under major topic headings in the questionnaire. A tabulation of the opinions of the respondents is included in each table.

The items listed in Table 2 are included for the purpose of building an understanding of the four computational processes. As the meaningful approach to teaching arithmetic developed, considerable emphasis was placed on the structure of our number system as it affects the mechanics of computation. Of equal importance is the ability to analyze social situations to determine the appropriate computational process. Proper presentation of the understandings in Table 2 can reveal the nature of our decimal number system and can enhance the analysis of problem-solving

TABLE 2
Definitions of Basic Computational Processes

	Manda- tory	Desir- able	To Be Omitted	No Re- sponse
Addition as describing the result of joining groups.	206	13	3	5
Subtraction as describing the result of removing part of a group	202	19	3	3
Subtraction as the result of comparing of two groups	196	23	2	6
Multiplication as describing the result of joining groups of equal size	207	13	3	4
Division as the breaking of groups into a known number of equal parts (partition)	202	20	2	3
Division as the description of finding how many equal groups are contained in a given group (measurement)	207	15	2	3

TABLE 3
Basic Principles Governing Fundamental Operations

	Manda- tory	Desir- able	To Be Omitted	No Re- sponse
Addition and subtraction, multiplication and division are inverse operations.	198	25	4	0
Only like quantities may be combined into single quantity by addition or subtraction	195	13	1	18
Numbers may be added or multiplied in any order without changing the value of sum or product (not true for subtraction and division).	198	25	3	1
To multiply a sum by a number, multiply each term by that number and add the resulting product (also true for dividing sum by a number)	172	43	4	8
To multiply a product by a number, multiply only one factor by that number (also true for dividing a product by a number)	159	47	8	13
The value of a fraction does not change if both numerator and denominator are multiplied or divided by the same number (other than zero)	208	11	0	8
Instead of dividing by a number you may multiply by its reciprocal or instead of multiplying by a number you may divide by its reciprocal	138	74	4	11

situations. The tabulations indicate strong support of this viewpoint by the profession.

The principles enumerated in Table 3 point out the interrelationships among the computational processes as well as some of the basic laws governing these processes. An analysis of responses to items in Table 3 shows strong support for the presentation of mathematics with a basic structure of a few principles rather than as a set of isolated mechanical manipulations. Some of the respondents stated a preference for using the basic laws as such, rather than developing the principles stated in the questionnaire.

The principles listed in Table 3 state some of the immediate consequences of the Laws of Commutation, Association, and Distribution involving addition and multiplication. Experience has shown that an understanding of these laws is necessary for teaching arithmetic effectively.

TABLE 4
Topics Encountered in Computation with Whole Numbers

	Mandatory	Desirable	To Be Omitted	No Response
Brief review of four fundamental operations . . .	175	39	5	8
Vocabulary (sum, difference, product, quotient, dividend, etc.)	196	31	0	0
Methods of checking each operation (including casting out nines)	125	84	8	10
Common types of errors including combination, carrying and position. Note that casting out nines does not detect positional errors	131	78	7	11
Estimation in all processes to check reasonableness of answer (this is not confined to whole numbers)	198	24	0	5
Short cuts (not confined to whole numbers) . . .	62	138	22	5

In a background mathematics course the time devoted to computation with whole numbers should be limited to review; diagnosis and remediation might be primary concerns. Certain facets of whole number computation must, however, be considered. An analysis of Table 4 reveals that the profession urges emphasis on review vocabulary and estimation in dealing with whole numbers; it favors, but does not consider mandatory, analyzing methods of checking, studying common types of errors, and learning short cuts in computation.

The topics in Table 4 cannot be presented in isolation. They are intimately related to the principles presented in Table 3, and it is impossible to discuss the basic situations of Table 2 without reviewing whole number operations. Certainly an examination of number systems in general and the historical beginnings of number (Table 9) cannot be divorced from work with whole numbers.

In training prospective teachers it is important to keep in mind that many students can perform the required operations but few can explain why they perform these operations as they do. In teaching the topics listed in Tables 4, 5, and 6, structure and meaning should be of major concern. Verbal problems should demonstrate that the choice of operations for solution depends upon the

TABLE 5
Topics Encountered in Computation with Common Fractions

	Manda- tory	Desir- able	To be Omitted	No Re- sponse
Definitions of a fraction	206	15	2	4
Adding and subtracting like fractions.....	204	15	1	7
Definition and methods for determining LCD....	197	24	2	4
Adding and subtracting unlike fractions.....	210	11	1	5
Multiplication of fractions.....	211	9	1	6
Rationalization of multiplication of fractions....	200	21	2	4
Division of fractions.....	205	16	0	6
Rationalization of inverting	187	29	6	5
Proper, improper fractions, mixed numbers, and compound or denominate numbers	198	21	2	6
Four fundamental operations with mixed and compound numbers.....	182	32	1	12

TABLE 6
Topics Encountered in Computation with Decimal Fractions

	Manda- tory	Desir- able	To Be Omitted	No Re- sponse
Definition of a decimal fraction emphasizing role of decimal point.....	213	7	0	7
Brief review of four fundamental operations....	181	37	2	7
Rationalization of placement of decimal point in fundamental operations.....	202	19	1	5
Changing from decimal to common fractions and vice versa.....	203	19	1	4
Complex fractions.....	118	87	8	14
Non-terminating decimal fractions.....	119	94	10	4
The rounding-off process as choice of least error..	161	57	3	6
The relative value of common or decimal frac- tions in various situations.....	158	62	2	5
Computation with approximate numbers.....	152	66	3	6

situations as outlined in Table 2 and not on whether the number is whole, common fraction or decimal fraction.

The properties and relative merits of common fractions and decimal fractions should help build greater understanding of fractions in general. Similarities and differences of computations with

common fractions, decimal fractions and whole numbers should be developed.

Table 5 and Table 6 show that the topics listed under each category of fractions are approved by a large majority of the respondents. There is less support for some facets of complex decimal fractions but most members of the profession consider them important.

The important place percent occupies in our every-day language as well as in business and industry makes it an essential topic in its own right. This viewpoint is supported by the overwhelming acceptance of the items in Table 7. Disagreement as to the appropriate approach to teaching percent is evidenced in the tabulations of the last three items. A valid interpretation of this section of Table 7 leads to the conclusion that each merits consideration and is appropriate in certain circumstances.

In considering the social applications of percent (Table 8) the respondents gave several additional suggestions. A common suggestion was the use of newspapers and periodicals for examples and illustrations of percent. Finance and athletics were listed as particularly fruitful applications of this concept. It is obvious in analyzing the results that most respondents consider social application of percent important but not as mandatory as other topics in the questionnaire.

TABLE 7
Mathematical Concepts Associated with Percent

	Mandatory	Desirable	To Be Omitted	No Response
Definition of percent	212	11	0	4
Changing from percent to decimal and common fractions	207	8	1	11
Changing to percent	209	12	2	4
Three fundamental situations using formula $p = br$	175	34	9	9
Three fundamental situations using unitary or one percent method	123	78	17	9
Relation of problems involving percent more than or less than to the three fundamental percent situations	185	22	13	7

TABLE 8
Social Applications of Percent

	Mandatory	Desirable	To Be Omitted	No Response
Mark-up, mark-down, discount, etc.....	159	60	4	4
Simple and compound interest.....	161	59	4	3
Insurance.....	122	89	8	8
Taxation.....	127	79	7	14
Installment buying.....	139	78	5	5
Social security.....	121	90	11	5

TABLE 9
Topics Pertaining to the Structure of the Hindu-Arabic Number System

	Mandatory	Desirable	To Be Omitted	No Response
Historical beginnings—other place systems (Babylonians and Egyptians).....	127	96	4	0
Comparison of our number system as a place system with some nonplace system as the Roman number system.....	136	82	6	3
Importance of zero in a place system.....	209	7	0	11
Counting in place systems other than base ten..	101	99	13	14
Changing numbers from other place systems to equivalent numbers in the base ten.....	90	112	24	1
Changing base ten numbers to equivalent numbers in other place systems with emphasis on binary and duodecimal systems.....	89	108	29	1
Emphasis on grouping and regrouping by performing fundamental operations in number systems (place systems) other than base ten..	78	113	33	3

Table 9 includes several topics related to the nature of the Hindu-Arabic decimal system of notation. An analysis of the tabulations suggests that members of the profession consider an understanding of the characteristics of our number system to be mandatory. On the other hand, dealing with the historical development of number and emphasizing number systems with bases other than ten are considered desirable but not mandatory.

Although the tabulation of Table 10 indicates a strong opinion

TABLE 10
Topics and Concepts Related to Algebra

	Manda- tory	Desir- able	To Be Omitted	No Re- sponse
Signed numbers.....	176	29	11	11
Four fundamental operations with monomials and simple polynomials.....	165	36	13	13
Linear equations.....	165	39	11	12
Factoring.....	134	63	18	12
Fractions.....	137	62	13	15
Fractional equations.....	128	68	15	16
Solutions of word problems.....	164	40	9	14
Graphing.....	159	45	10	13
Exponents.....	127	75	12	13
Logarithms.....	60	109	41	17
Quadratic equations.....	73	97	43	14

in favor of some algebraic concepts, a sizable group considers algebra unnecessary. Several respondents commented that an understanding of the items listed in Table 10 would foster understanding of arithmetic. Strong sentiment was expressed in favor of including algebra for its cultural and aesthetic value.

The substantial vote for the three items in Table 11 indicates overwhelming support of the comments pertaining to Table 10.

TABLE 11
The Methodology of Relating Algebra and Arithmetic

	Manda- tory	Desir- able	To Be Omitted	No Re- sponse
Algebra as an extension of arithmetic (essen- tially the same principles listed under arith- metic govern the operations in algebra).....	181	24	7	15
Algebra as a language. Solution of word prob- lems requires the ability to replace words by algebraic symbols. It can be very helpful to translate from symbols into words also.....	182	23	7	15
Algebra as a tool for generalization. To solve an arithmetic problem solves only one problem. By using letters for numbers, one may solve all such problems.....	177	27	8	15

TABLE 12
Concepts of Size Stemming from Geometry

	Manda- tory	Desir- able	To Be Omitted	No Re- sponse
Common formulas for area	183	26	6	12
Common formulas for volume	177	32	6	12

TABLE 13
Concepts of Shape Stemming from Geometry

	Manda- tory	Desir- able	To Be Omitted	No Re- sponse
Definitions and concepts	166	42	5	14
Intuitive and informal conclusions	150	54	8	15

An essential ingredient in building algebraic concepts is the emphasis upon meaning and understanding.

Concepts of size and shape stemming from geometry (Tables 12 and 13) were regarded as desirable for cultural and aesthetic purposes. Comments were made that the influence of geometry in design and art is as important as the study of basic properties of common geometric configurations.

The questionnaire contained a group of items pertaining to the nature of logic; these are enumerated in Table 14. Several returns expressed the opinion that a series of isolated lessons given to logic with no cross reference to other aspects of mathematics

TABLE 14
The Nature of Logic in Mathematics

	Manda- tory	Desir- able	To Be Omitted	No Re- sponse
Inductive reasoning as reasoning from specific to general	137	71	7	12
Deductive reasoning as necessary consequence of premises	137	73	7	10
Distinction between truth and validity	122	85	13	7
The nature of mathematical proof	117	84	17	9
Non-mathematical applications of logic	78	104	32	13

would be ineffective. Some respondents felt that logic should appear early in the course to demonstrate its significance in several areas. An analysis of Table 14 reveals that a large segment of the profession questions the necessity of logic as a part of a background course in mathematics for teachers.

There was less support of topics in trigonometry than in any other phase of mathematics. The items in Table 15 received support primarily from the standpoint of their cultural and aesthetic value in mathematics.

Widespread acceptance of teaching for meaning and understanding in mathematics is reflected in Table 16. Apparently the

TABLE 15
Concepts and Topics Stemming from Trigonometry

	Manda- tory	Desir- able	To Be Omitted	No Re- sponse
Definition of sine, cosine, and tangent in right triangle.	97	90	27	13
Solution of right triangle.	107	85	23	12
Simple applications of trigonometry to indirect measurement.	102	90	23	12

TABLE 16
Philosophy Underlying the Presentation of Mathematical Concepts

	Manda- tory	Desir- able	To Be Omitted	No Re- sponse
General knowledge.	162	44	5	16
Ability to compute.	162	46	3	16
Understanding and concepts.	169	39	3	16
Alteration of course on basis of pre-test.	136	64	6	21
Remedial work for weak students.	130	69	9	19
Requirement of additional work or project for better students.	103	94	11	19
Required outside reading for familiarization with literature available to non-specialists in mathematics.	81	112	15	19
Emphasis on discovery to the extent that time permits.	126	82	5	14
Emphasis on thought patterns and methods.	127	73	13	14

profession at large believes that a background course in mathematics for prospective teachers should be geared to the individual. Various techniques for achieving this goal are presented in Table 16. It is interesting to note that the technique of requiring additional readings, non-technical in nature, is considered less important than other teaching techniques.

RECOMMENDATIONS

The data assembled in this chapter show that the training of teachers of arithmetic leaves much to be desired. The evidence described in the first section deals with the meager background, lack of understanding, poor attitude, and inadequate working conditions of elementary teachers. The next section analyzes the quantity and quality of mathematics training required of elementary teachers. Then the results of a questionnaire proposed for a background course in mathematics are presented. On the strength of evidence summarized in this chapter, the writers make the following recommendations for the improvement of arithmetic instruction in the elementary grades:

1. A teacher should not be certified to teach in the elementary school who has not completed the work for a bachelor's degree.
2. Certain specialized training must be required of elementary school teachers. Teachers at all grade levels should have at least a minimum of course content in mathematics and methods of teaching mathematics.
3. Each student admitted to the elementary teacher training program should be required to take a minimum of six semester hours of background mathematics. The tabulations and discussion in this chapter provide a check list of topics for organizing such a course.
4. The writers do not advocate the traditional algebra or geometry sequence of courses as suitable training for elementary school teachers. The lists of topics in Tables 2 through 15 are equally unsatisfactory if presented for the purpose of developing mechanical skills. Developing of principles and understanding of concepts are essential elements of an adequate background course.
5. The order of topics as they appear in Tables 2 through 15

is not necessarily the most desirable organization of a background mathematics course. A variety of acceptable sequences is possible.

6. Teachers of arithmetic in grades seven and eight should complete a minor in college mathematics.

7. All elementary school teachers should have a course in the teaching of arithmetic which should follow the background sequence in mathematics.

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Selected Annotated Bibliography

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IF THE CURRENT literature dealing with the teaching of arithmetic is taken as a criterion, then the outlook for school arithmetic is indeed promising. During the past 10 or 15 years, arithmetic teaching has been influenced by several significant trends: (a) the enthusiastic acceptance of the meaning theory of learning arithmetic, (b) a more searching study of problem solving in quantitative situations, (c) new techniques for evaluating the outcomes of instruction in arithmetic, and (d) a more general use of manipulative devices and concrete materials, over and above the customary visual aids. These trends are clearly reflected in the literature. A fifth trend should perhaps also be noted, although it has not yet found its way generally into the literature. In the training of teachers, attention is gradually being given to the logical foundations of arithmetic—the theoretical structure of arithmetic as a rigorous discipline. While it may be too early to tell, it would

seem as though this consideration, too, will eventually exert a salutary influence upon the teaching of arithmetic.

The present bibliography covers a period of some 16 years—from 1941, the year in which the National Council's last Yearbook on Arithmetic was published, to the middle of the year 1958. Anything like a complete listing for such a long period within the space limitations of this volume was manifestly out of the question; the literature is much too extensive. Thus the chief problem was one of selection. To decide which articles and which studies would be of greatest interest and maximum value was no easy task. The following factors served in large measure as criteria for selection: (a) the intrinsic importance of the problem or topic, (b) the cogency of the findings and conclusions, (c) the possible implications for educational practice, (d) the merits of the procedure used, or the professional standing of the author, and (e) the date of publication. Frequently, when an author returned to the same problem at a later date, we have arbitrarily chosen his more recent contribution and excluded the earlier reference. Annotation was also a matter of some concern. Lengthy annotations, while often desirable, are not possible when space is limited. Some papers are rather clearly defined by their titles; others are not. Moreover, the usefulness of an annotation depends largely upon the reader's purpose. The attempt here has been to suggest the nature and scope of the contents, with some indication, if possible, of the author's conclusions or the significance of the discussion. If this mode of operation seems somewhat arbitrary, it should be remembered that all bibliographic searching is to some extent unavoidably subjective. Much depends upon the judgment and experience of the compiler, whose responsibility it is to produce a list that is professionally acceptable, and at the same time both adequate and useful. It is hoped that this bibliography does not fall too wide of the desired goal.

ORGANIZATION

For the reader's convenience, the bibliography has been organized in accordance with the following divisions:

1. *Current Practice, Trends, Research*
2. *Psychology of Learning Arithmetic*

3. *Meaning Theory of Teaching Arithmetic*
4. *General Methodology and Improvement of Instruction*
5. *Problem Solving*
6. *Manipulative Devices, Visual Aids, Concrete Materials*
7. *Curriculum Organization*
8. *Instructional Material: Kindergarten and Grades 1 and 2*
9. *Instructional Material: Grades 3 through 9*
10. *Measurement and Evaluation*
11. *Numeration: Addition and Subtraction of Whole Numbers*
12. *Multiplication and Division of Whole Numbers*
13. *Fractions, Decimals, and Percents*
14. *Preparation of Teachers*
15. *Books on the Teaching of Arithmetic*
16. *Books on Arithmetic Subject Matter*

1. CURRENT PRACTICE, TRENDS, RESEARCH

ARTHUR, LEE E. "Diagnosis of Disabilities in Arithmetic Essentials." *Mathematics Teacher* 43: 197-202; May 1950.

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BUCKINGHAM, B. R. "The Social Point of View in Arithmetic." *The Teaching of Arithmetic*. Fiftieth Yearbook, Part II, National Society for the Study of Education. Chicago: University of Chicago Press, 1951. Chapter 14, p. 269-81.

Growth in arithmetic is cumulative; social institutions have an impact on arithmetic; skills vs. content or processes vs. concepts.

BUSWELL, G. T. "Needed Research on Arithmetic." *The Teaching of Arithmetic*. Fiftieth Yearbook, Part II, National Society for the Study of Education. Chicago: University of Chicago Press, 1951. Chapter 15, p. 282-97.

Outlines 21 specific proposals for future research in arithmetic teaching and learning.

CLARK, JOHN R. "Issues in Teaching Arithmetic." *Teachers College Record* 52: 205-12; January 1951.

Informal discussion of current thinking concerning important problems in the pedagogy of arithmetic.

DEPARTMENT OF ELEMENTARY SCHOOL PRINCIPALS. "Arithmetic in the Modern Elementary School." *National Elementary Principal* 30: 1-56; October 1950.

An overall survey of contemporary conditions.

- DUTTON, WILBUR H. "Attitudes of Junior High School Pupils Toward Arithmetic." *School Review* 64: 18-22; January 1956.

Findings according to an attitude scale devised by the author.

- DYER, HENRY S.; KALIN, ROBERT; AND LORD, FREDERIC M. *Problems in Mathematical Education*. Princeton, N. J.: Educational Testing Service, 1956. 50 p.

A survey of current issues and criticisms embracing both elementary and secondary levels.

- EADS, LAURA K. "Ten Years of Meaningful Arithmetic in New York City." *Arithmetic Teacher* 2: 142-47; December 1955.

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- FEHR, HOWARD, F. "A Philosophy of Arithmetic Instruction." *Arithmetic Teacher* 2: 27-32; April 1955.

An eclectic philosophy based upon modern theories of learning: the ultimate goal is functional problem-solving ability in quantitative situations.

- FEHR, HOWARD F. "Present Research in the Teaching of Arithmetic." *Teachers College Record* 52: 11-23; October 1950.

Highlights of research on problem solving, meaning, social utility, transfer of training, evaluation and sensory aids; needed research; bibliography.

- GLENNON, VINCENT J., AND HUNNICUTT, C. W. *What Does Research Say About Arithmetic?* Washington, D. C.: Association for Supervision and Curriculum Development, a department of the National Education Association, 1958. 77 p.

Summary of findings and conclusions relating to problems dealing with elementary school mathematics.

- HABEL, E. A. "Deficiencies of College Freshmen in Arithmetic: Diagnosis and Remedy." *School Science and Mathematics* 50: 480-84; June 1950.

A review of the literature; bibliography.

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UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION. *Introduction to Mathematics in Primary Schools*. Paris: International Bureau of Education, Publication No. 121, 1950. 247 p.

Reports of current practices in the teaching of arithmetic in nearly 50 foreign countries.

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An honest attempt to rethink the implications of modern learning theory for the teaching of arithmetic.

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- WRIGHTSTONE, J. WAYNE. "Influence of Research on Instruction in Arithmetic." *Mathematics Teacher* 45: 187-92; March 1952.

Presents the implications of 27 studies and summaries.

2. PSYCHOLOGY OF LEARNING ARITHMETIC

- ANDERSON, G. LESTER. "Quantitative Thinking as Developed under Connectionist and Field Theories of Learning." *Learning Theory in School Situations*. University of Minnesota Studies in Education, No. 2. Minneapolis: University of Minnesota Press, 1949. p. 40-73.

Experimental study comparing two methods of teaching: (1) emphasizing understanding and generalization, and (2) emphasizing specific facts and skills.

- BROWNELL, WILLIAM A. "The Progressive Nature of Learning in Mathematics." *Mathematics Teacher* 37: 147-57; April 1944.

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- WHEAT, HARRY G. "The Nature and Sequences of Learning Activities in Arithmetic." *The Teaching of Arithmetic*. Fiftieth Yearbook, Part II, National Society for the Study of Education. Chicago: University of Chicago Press, 1951. Chapter 3, p. 22-52.

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3. MEANING THEORY OF TEACHING ARITHMETIC

- BROWNELL, WILLIAM A. "The Place of Meaning in the Teaching of Arithmetic." *Elementary School Journal* 47: 256-65; January 1947.

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BROWNELL, WILLIAM A. "When is Arithmetic Meaningful?" *Journal of Educational Research* 38: 481-98; March 1945.

Reasons for teaching meaning as well as skill: (1) understanding makes for functional use, (2) meanings facilitate learning, (3) meanings increase likelihood of transfer, (4) meaning makes for greater retention and easier rehabilitation.

BURCH, ROBERT L. "Skills Instruction in Arithmetic." *National Elementary Principal* 29: 25-33; December 1949.

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DAWSON, DANIEL T., AND RUDELL, ARDEN K. "The Case for the Meaning Theory in Teaching Arithmetic." *Elementary School Journal* 55: 393-99; March 1955.

Summary of conclusions of five important studies of one aspect or another of the "meaning theory."

HENDRIX, GERTRUDE. "Prerequisite to Meaning." *Mathematics Teacher* 43: 334-39; November 1950.

Meaning is achieved when the symbols which represent previously achieved concepts are learned.

HICKERSON, J. ALLEN. "The Semantics and Grammar of Arithmetic Language." *Arithmetic Teacher* 2: 12-16; February 1955.

Another approach to "meaning" by having pupils discover relationships and formulate them verbally in their own language.

JOHNSON, J. T. "What Do We Mean by Meaning in Arithmetic?" *Mathematics Teacher* 41: 362-67; December 1948.

Distinguishes among (1) structural meanings, (2) functional meanings, and (3) rationalization of processes.

McSWAIN, ELDRIGE T. "Discovering Meanings in Arithmetic." *Childhood Education* 26: 267-71; February 1950.

Stresses the role of discovery in the attainment of meaning.

MORTON, ROBERT L. "Language and Meaning in Arithmetic." *Educational Research Bulletin* 34: 197-204; November 1955.

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RIESS, ANITA. "The Meaning of 'Meaningful' Teaching of Arithmetic." *Elementary School Journal* 45: 23-32; September 1944.

The psychological development of the two basic concepts of numbers: (1) the naming function and (2) the numbering and grouping function.

VAN ENGEN, HENRY. "An Analysis of Meaning in Arithmetic." *Elementary School Journal* 49: 321-29, 395-400; February-March 1949.

An operational viewpoint; the learner must see and act before he is presented with the symbol that represents the act.

WEAVER, J. FRED. "Misconceptions about Rationalization in Arithmetic." *Mathematics Teacher* 44: 377-81; October 1951.

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4. GENERAL METHODOLOGY AND IMPROVEMENT OF INSTRUCTION

BEATTY, MRS. LESLIE S. "Re-orienting to the Teaching of Arithmetic." *Childhood Education* 26: 272-78; February 1950.

Summary of research.

BROWNELL, WILLIAM A. "Arithmetic Readiness as a Practical Classroom Concept." *Elementary School Journal* 52: 15-22; September 1951.

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GIBB, E. GLENADINE. "A Review of a Decade of Experimental Studies Which Compared Methods of Teaching Arithmetic." *Journal of Educational Research* 46: 603-608; April 1953.

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GUILER, WALTER S., AND EDWARDS, VERNON. "Experimental Study of Methods of Instruction in Computational Arithmetic." *Elementary School Journal* 43: 353-60; February 1943.

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HARDING, LOWRY W., AND BRYANT, INEZ. "An Experimental Comparison of Drill and Direct Experience in Arithmetic

Learning in a Fourth Grade." *Journal of Educational Research* 37: 321-37; January 1944.

Direct, first-hand-experience method produces greater improvement in reasoning than drill procedures and vicarious experiences.

HARVEY, LOIS F. "Improving Arithmetic Skills by Testing and Reteaching." *Elementary School Journal* 53: 402-409; March 1953.

An analysis of errors made in multiplication; discussion of effectiveness of diagnostic tests and remedial instruction.

HOEL, LESTA. "An Experiment in Clinical Procedures for Arithmetic." *Emerging Practices in Mathematics Education*. Twenty-Second Yearbook. Washington, D. C.: National Council of Teachers of Mathematics, a department of the National Education Association, 1954. p. 222-32.

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HOWARD, CHARLES F. "Three Methods of Teaching Arithmetic." *California Journal of Educational Research* 1: 25-29; January 1950.

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KNIPP, MINNIE B. "An Investigation of Experimental Studies which Compare Methods of Teaching Arithmetic." *Journal of Experimental Education* 13: 23-30; September 1944.

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LE BARON, WALTER A. "A Study of Teachers' Opinions in Methods of Teaching Arithmetic in the Elementary School." *Journal of Educational Research* 43: 1-9; September 1949.

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SAUBLE, IRENE. "Development of Ability to Estimate and to Compute Mentally." *Arithmetic Teacher* 2: 33-39; April 1955.

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WEAVER, J. FRED. "Differentiated Instruction in Arithmetic: An Overview and a Promising Trend." *Education* 74: 300-305; January 1954.

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WEAVER, J. FRED. "Big Dividends from Little Interviews." *Arithmetic Teacher* 2: 40-47; April 1955.

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5. PROBLEM SOLVING

BOWERS, JOAN E. "Procedure to Strengthen Ability to Solve Arithmetical Problems." *School Science and Mathematics* 57: 485-93; June 1957.

Gives a number of substantial, specific suggestions, based upon psychological as well as mathematical considerations.

BURCH, ROBERT L. "Formal Analysis as a Problem-Solving Procedure." *Journal of Education* 136: 44-47, 64; November 1953.

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- CRONBACH, LEE J. "The Meanings of Problems." *Arithmetic 1948*. Supplementary Educational Monographs, No. 66. Chicago: University of Chicago Press, 1948. p. 32-43.

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- HALL, JACK V. "Mental Arithmetic: Misunderstood Terms and Meanings." *Elementary School Journal* 54: 349-53; February 1954.

Discussion of problems to be solved mentally in fifth and sixth grades.

- HALL, JACK V. "Solving Verbal Arithmetic Problems Without Pencil and Paper." *Elementary School Journal* 48: 212-17; December 1947.

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- HANSEN, CARL W. "Factors Associated with Successful Achievement in Problem Solving in Sixth-Grade Arithmetic." *Journal of Educational Research* 38: 111-18; October 1944.

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- JOHNSON, JOHN T. "On the Nature of Problem Solving in Arithmetic." *Journal of Educational Research* 43: 110-15; October 1949.

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- KLUGMAN, SAMUEL F. "Cooperative versus Individual Efficiency in Problem Solving." *Journal of Educational Psychology* 35: 91-100; February 1944.

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- PETTY, OLAN. "Non-Pencil-and-Paper Solution of Problems." *Arithmetic Teacher* 3: 229-35; December 1956.

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- SPITZER, HERBERT F., AND FLOURNOY, FRANCES. "Developing Facility in Solving Verbal Problems." *Arithmetic Teacher* 3: 177-82; November 1956.

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- SUTHERLAND, ETHEL. *One-step Problem Patterns and their Relation to Problem Solving in Arithmetic*. Contributions to Education, No. 925. New York: Teachers College, Columbia University, 1947. 170 p. (Summary in *Teachers College Record* 49: 492; April 1948.)

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- SUTHERLAND, JOHN. "Investigation into Some Aspects of Problem Solving in Arithmetic." *British Journal of Educational Psychology* 11: 215-22; November 1941. 12: 35-46; February 1942.

A statistical study of 350 children, ages 10-11; identifies at least three significant factors important to success.

TREACY, JOHN P. "The Relationship of Reading Skills to the Ability to Solve Arithmetic Problems." *Journal of Educational Research* 38: 86-96; October 1944.

Study of 274 seventh-grade pupils; difficulty with problem solving is related to ability to retain significant facts and to vocabulary difficulties.

6. MANIPULATIVE DEVICES, VISUAL AIDS, CONCRETE MATERIALS

ANDERSON, GEORGE R. "Visual-Tactual Devices and Their Efficiency: An Experiment in Grade Eight." *Arithmetic Teacher* 4: 196-203; November 1957.

An attempt to assess the value of multi-sensory aids, chiefly geometric in character.

FEHR, HOWARD F.; McMEEN, GEORGE; AND SOBEL, MAX. "Using Hand-Operated Computing Machines in Learning Arithmetic." *Arithmetic Teacher* 3: 145-50; October 1956.

An experimental study involving fifth-grade children.

FLEWELLING, ROBERT W. "The Abacus as an Arithmetic Teaching Device." *Arithmetic Teacher* 2: 107-11; November 1955.

Construction and use of a simple abacus; provides insight into our number system and our modes of calculating.

GROSSNICKLE, FOSTER E. "Materials for Grade IV Through Grade VI." *Emerging Practices in Mathematics Education*. Twenty-Second Yearbook. Washington, D. C.: National Council of Teachers of Mathematics, a department of the National Education Association, 1954. p. 131-34.

Manipulative materials appropriate for these grade levels; if and when pupils can generalize with symbols, the manipulative materials should be dispensed with.

GROSSNICKLE, FOSTER E.; JUNG, CHARLOTTE; AND METZNER, WILLIAM. "Instructional Materials for Teaching Arithmetic." *The Teaching of Arithmetic*. Fiftieth Yearbook, Part II, National Society for the Study of Education. Chicago: University of Chicago Press, 1951. Chapter 9, p. 155-85.

Illustrates the use of real experiences, manipulative materials, pictorial materials and symbolic materials at various grade levels; also gives

lists of commercially available manipulative and pictorial materials, as well as slides and films.

- GROSSNICKLE, FOSTER E., AND METZNER, WILLIAM. *Use of Visual Aids in the Teaching of Arithmetic*. Brooklyn, N. Y.: Rambler Press, 1950. 58 p.

Discusses the uses of films, filmstrips, and slides in teaching arithmetic; gives list of films, filmstrips, and slides by title and source, together with description and appraisal.

- HEARD, IDA MAE. "Handmade Materials for Teaching Arithmetic—Materials for Kindergarten Through Grade III." *Emerging Practices in Mathematics Education*. Twenty-Second Yearbook. Washington, D. C.: National Council of Teachers of Mathematics, a department of the National Education Association, 1954. p. 126-31.

Describes the use of (a) the string of 100 spools and (b) nine separate racks of spools to expedite the learning of addition and subtraction facts.

- LAZAR, NATHAN. "A Device for Teaching Concepts and Operations Relating to Integers and Fractions." *Arithmetic 1949*. Supplementary Educational Monographs, No. 70. Chicago: University of Chicago Press, 1949. p. 87-100.

A unique device consisting of two parts: a counting frame and a modified abacus.

- MACLATCHY, JOSEPHINE H. "Seeing and Understanding in Number." *Elementary School Journal* 45: 144-52; November 1944.

Describes the use of "markers" in teaching numbers.

- MALONEY, JOHN P. "Arithmetic at the Primary Level." *Arithmetic Teacher* 4: 112-18; April 1957.

An analysis of numbers and of the number system through objective development by means of visual and manipulative materials.

- MOTYKA, AGNES L. "Learning Aids in Arithmetic." *National Elementary Principal* 30: 34-41; October 1950.

A review of available learning aids.

- SCHOTT, ANDREW F. "New Tools, Methods for their Use, and a New Curriculum in Arithmetic." *Arithmetic Teacher* 4: 204-209; November 1957.

Report of preliminary research on the use of the counting frame, the abacus, arithmetic kits, calculating machines and other non-verbal tools.

SPITZER, HERBERT F. "The Abacus in the Teaching of Arithmetic." *Elementary School Journal* 42: 448-51; February 1942.

Usefulness of the abacus in exploring meanings of numbers in our decimal system of notation.

SPITZER, HERBERT F. "Device as an Aid in Teaching the Idea of Tens." *School Science and Mathematics* 42: 65-68; January 1942.

The "ten block" device for the understanding of two-place numbers (and larger numbers).

STERN, CATHERINE. "The Concrete Devices of Structural Arithmetic." *Arithmetic Teacher* 5: 119-30; April 1958.

Shows how structurally adequate materials can make abstract numbers and number relations visible and concrete.

SUELTZ, BEN A. "Counting Devices and Their Uses." *Arithmetic Teacher* 1: 25-30; February 1954.

Construction and use of counting frames, the abacus, counter board, magnetic board, flannel board, number stick, number pockets, number charts, etc.

UNIVERSITY OF ILLINOIS COMMITTEE ON SCHOOL MATHEMATICS. "Arithmetic with Frames." *Arithmetic Teacher* 4: 119-24; April 1957.

A convenient and intriguing device for stimulating thinking and discovery; will bear watching.

7. CURRICULUM ORGANIZATION

BERNADETTE, SISTER MARY. "An Elementary Supervisor Looks at Arithmetic." *School Science and Mathematics* 50: 445-53; June 1950.

A plan for the satisfactory teaching of any complete unit of work in arithmetic.

FAWCETT, HAROLD P. "A Unified and Continuous Program in Mathematics." *School Science and Mathematics* 50: 342-48; May 1950.

Suggests that the concepts of number, measurement, relationship, proof, operation, and symbolism grow in meaning as mathematics is learned.

GATES, LUCILE G. "Spiral Development of Arithmetic." *School Science and Mathematics* 49: 273-80; April 1949.

Describes a step-by-step spiral development of division as typical of a spiral organization of arithmetic.

HORN, ERNEST. "Arithmetic in the Elementary-School Curriculum." *The Teaching of Arithmetic*. Fiftieth Yearbook, Part II, National Society for the Study of Education. Chicago: University of Chicago Press, 1951. Chapter 2, p. 6-21.

A plea for greater emphasis upon arithmetical and quantitative concepts; discussion of interrelationships between arithmetic and other subjects.

MORTON, JOHN A. "A Study of Children's Mathematical Interest Questions as a Clue to Grade Placement of Arithmetic Topics." *Journal of Educational Psychology* 37: 295-315; May 1946.

Analysis of questions asked by 4250 children (grades 1-8) on aviation; implications for curriculum organization in arithmetic.

MORTON, ROBERT L. "The Place of Arithmetic in Various Types of Elementary-School Curriculums." *Arithmetic 1949*. Supplementary Educational Monographs, No. 70. Chicago: University of Chicago Press, 1949. p. 1-20.

Analyzes the question on the basis of three criteria: *logical*, *social* and *psychological*.

PASSEHL, GEORGE. "Teaching Arithmetic Through Activity Units." *Peabody Journal of Education* 27: 148-52; November 1949.

Five social units used for teaching the fundamental operations with decimals, as well as multiplication and division of common fractions.

ULRICH, ROBERT P. "Grade Placement of Computational Topics in Arithmetic." *Elementary School Journal* 42: 50-59; September 1941.

Analysis of eight textbook series, grades 3-6.

WILLIAMS, CATHERINE M. "Arithmetic Learning in an Experience

Curriculum." *Educational Research Bulletin* 28: 154-68; September 1949.

Study of nine successive sixth-grade groups; tests showed that children in the experience curriculum scored higher in arithmetic reasoning than in abstract examples.

WILSON, GUY M. "The Social Utility Theory as Applied to Arithmetic, Its Research Basis, and Some of Its Implications." *Journal of Educational Research* 41: 321-37; January 1948.

A stimulating and provocative discussion.

8. INSTRUCTIONAL MATERIAL: KINDERGARTEN AND GRADES 1 AND 2

ADAMS, OLGA. "Arithmetic Readiness in the Primary Grades." *Arithmetic* 1947. Supplementary Educational Monographs, No. 63. Chicago: University of Chicago Press, 1947. p. 10-16.

Readiness in the early years of a child's life can be built around counting, measuring, fundamental processes, fractions, and telling time.

CLARK, EILEEN. "Number Experiences of Three-Year-Olds." *Childhood Education* 26: 247-50; February 1950.

Report on the numerical vocabulary of nursery-school children.

DEANS, EDWINA. "The Practical Aspects of Number Work at the Primary-Grade Level." *Arithmetic* 1947. Supplementary Educational Monographs, No. 63. Chicago: University of Chicago Press, 1947. p. 17-22.

Incidental versus planned number experience; social values of number; concrete number materials; grouping children for number work.

GRIME, HERSCHEL E. "Adapting the Curriculum in Primary Arithmetic to the Abilities of Children." *Mathematics Teacher* 43: 242-44; October 1950.

Discussion of ways and means of providing for individual differences.

KOENKER, ROBERT H. "Arithmetic Readiness for the Primary Grades." *Arithmetic* 1949. Supplementary Educational Monographs, No. 70. Chicago: University of Chicago Press, 1949. p. 26-34.

Considers five questions relevant to arithmetic readiness of young children.

- MACLATCHY, JOSEPHINE H. "The Pre-School Child's Familiarity with Measurement." *Education* 71: 479-82; April 1951.

Based on an inventory test administered to three-, four-, and five-year-olds; reveals insight into concepts and knowledge of measuring instruments.

- MACLATCHY, JOSEPHINE H. "A Test of the Pre-School Child's Familiarity with Measurement." *Educational Research Bulletin* 29: 207-208, 222-23; November 1950.

Description of an inventory test designed for six-year-olds.

- MOTT, SINA M. "Number Concepts of Small Children." *Mathematics Teacher* 38: 291-301; November 1945.

Investigation of number concepts of 4- and 5-year-old children in nursery school and kindergarten.

- MOTT, SINA M., AND MARTIN, MARY ELIZABETH. "Do First Graders Retain Number Concepts Learned in Kindergarten?" *Mathematics Teacher* 40: 75-78; February 1947.

Children carry over into first grade number experiences previously learned with the exception of rote counting from 1 to 100.

- STOTLAR, CAROLYN. "Arithmetic Concepts of Preschool Children." *Elementary School Journal* 46: 342-45; February 1946.

Children of ages 50 to 69 months have a definite awareness and some understanding of numbers before entering school.

- SWENSON, ESTHER J. "Arithmetic for Preschool and Primary-Grade Children." *The Teaching of Arithmetic*. Fiftieth Yearbook, Part II, National Society for the Study of Education. Chicago: University of Chicago Press, 1951. Chapter 4, p. 53-75.

Child growth, readiness, and the development of the number concept in the child's mind.

- WILBURN, D. BANKS. "Methods of Self-Instruction for Learning Arithmetic." *Arithmetic* 1949. Supplementary Educational Monographs, No. 70. Chicago: University of Chicago Press, 1949. p. 35-43.

Experimental results indicate that pupils in the early grades can learn by methods of self-instruction.

9. INSTRUCTIONAL MATERIAL: GRADES 3 THROUGH 9

BERNSTEIN, ALLEN L. "A Study of Remedial Arithmetic Conducted with Ninth-Grade Students." *School Science and Mathematics* 56: 25-31, January; 429-37, June 1956.

A 2-year study of diagnostic procedures and a remedial program.

BRUECKNER, LEO JOHN. "Arithmetic in Elementary and Junior High Schools." *Arithmetic* 1947. Supplementary Educational Monographs, No. 63. Chicago: University of Chicago Press, 1947. p. 1-9.

Integration of the mathematical and social phases of arithmetic can be achieved on the basis of five major considerations.

CHASE, W. LINWOOD. "Subject Preferences of Fifth-Grade Children." *Elementary School Journal* 50: 204-11; December 1949.

A survey of 13,383 fifth-grade children in New England and 2350 fifth-graders in a Southwestern city.

FLOURNOY, MARY FRANCES. "The Effectiveness of Instruction in Mental Arithmetic." *Elementary School Journal* 55: 148-53; November 1954.

Data based on a study of sixth-grade pupils.

GLENNON, VINCENT J. "Testing Meanings in Arithmetic." *Arithmetic* 1949. Supplementary Educational Monographs, No. 70. Chicago: University of Chicago Press, 1949. p. 64-74.

A pioneer study of the extent of growth and mastery of certain basic mathematical understandings possessed by learners in grades 7, 8, 9, 12, college freshmen, teachers in training, and teachers in service.

HARTUNG, MAURICE L. "Major Instructional Problems in Arithmetic in the Middle Grades." *Arithmetic* 1949. Supplementary Educational Monographs, No. 70. Chicago: University of Chicago Press, 1949. p. 80-86.

General discussion of the variable factors influencing pupil achievement in arithmetic in grades 3-6.

HARTUNG, MAURICE L. "Major Instructional Problems in Arithmetic in the Middle Grades." *Elementary School Journal* 50: 86-91; October 1949.

Discusses the relative balance between emphasis on meaning and emphasis on drill.

- HOUSTON, LUCILLE. "Articulating Junior High Mathematics with Elementary Arithmetic." *School Science and Mathematics* 51: 117-21; February 1951.

The role of measurement, ratio, and percentage in the 7th and 8th year programs.

- KINSELLA, JOHN J. "The Adolescent and Arithmetic." *School Science and Mathematics* 50: 119-24; February 1950.

Implications of personality and developmental traits for teaching arithmetic to junior-high-school pupils.

- MACLATCHY, JOSEPHINE H., AND HUMMEL, EUGENIE. "Arithmetic with Understanding." *Educational Research Bulletin* 21: 227-38, 246; November 1942.

Procedures used in meaningful teaching with a group of children in grades 3 and 4.

- THIELE, C. L. "Arithmetic in the Middle Grades." *The Teaching of Arithmetic*. Fiftieth Yearbook, Part II, National Society for the Study of Education. Chicago: University of Chicago Press, 1951. Chapter 5, p. 76-102.

A plea for greater emphasis upon meaning and discovery in the arithmetic of the middle grades; also, for less haste in moving from the concrete to the abstract.

- VAN ENGEN, HENRY. "Arithmetic in the Junior-Senior High School." *The Teaching of Arithmetic*. Fiftieth Yearbook, Part II, National Society for the Study of Education. Chicago: University of Chicago Press, 1951. Chapter 6, p. 103-19.

Suggests that junior-senior arithmetic should emphasize measurement, comparisons, operations, dependence, and problem solving.

- WILLCUTT, GLADYS M. "Classroom Experiences with Pupil Participation in Teaching Arithmetic." *Arithmetic 1949*. Supplementary Educational Monographs, No. 70. Chicago: University of Chicago Press, 1949. p. 44-54.

Report on experiences with seventh-grade pupils when learning informal geometry.

10. MEASUREMENT AND EVALUATION

- BOUCHARD, JOHN B. "An Exploratory Investigation of the Effect of Certain Selected Factors upon Performance of Sixth-Grade

Children in Arithmetic." *Journal of Experimental Education* 20: 105-12; September 1951.

Study of the effect of having received some knowledge of the testing situation upon subsequent test performance.

MOYER, HAVERLY O. "Testing the Attainment of the Broader Objectives of Arithmetic." *Arithmetic Teacher* 3: 66-70; March 1956.

A unique and original device for evaluation of learning which combines learning and testing.

OLANDER, HERBERT T.; VAN WAGENEN, M. J.; AND BISHOP, HELEN MIRIAM. "Predicting Arithmetic Achievement." *Journal of Educational Research* 43: 66-73; September 1949.

Use of two specially constructed arithmetic scales: (1) to measure quantitative information, and (2) to evaluate pupils' ability to recognize relationships.

SPACHE, GEORGE. "Tests of Abilities in Arithmetic Reasoning." *Elementary School Journal* 47: 442-45; April 1947.

Construction of a test to measure five specific aspects of problem solving; concludes that reading skill is definitely related to success in problem solving.

SPITZER, HERBERT F. "Procedures and Techniques for Evaluating the Outcomes of Instruction in Arithmetic." *Arithmetic 1948*. Supplementary Educational Monographs, No. 66. Chicago: University of Chicago Press, 1948. p. 15-25. Also, *Elementary School Journal* 49: 21-31; September 1948.

Proposes a number of innovations in evaluating pupil achievement in arithmetic which have since become more widely known and generally accepted.

SPITZER, HERBERT F. "Testing Instruments and Practices in Relation to Present Concepts of Teaching Arithmetic." *The Teaching of Arithmetic*. Fiftieth Yearbook, Part II, National Society for the Study of Education. Chicago: University of Chicago Press, 1951. Chapter 10, p. 186-202.

Calls attention to inadequacies of present testing practices, with recommendations for improvement; gives partial list of published tests.

STORM, W. B. "Arithmetical Meanings That Should Be Tested."

Arithmetic 1948. Supplementary Educational Monographs, No. 66. Chicago: University of Chicago Press, 1948. p. 26-31.

A list of 60 or 70 specific items of significance which should be tested.

SUELTZ, BEN A. "Measuring the Newer Aspects of Functional Arithmetic." *Improving the Program in Arithmetic*. (G. T. BUSWELL, Editor.) Chicago: University of Chicago Press, 1946. p. 27-34.

New evaluation techniques; the value of supplementing printed tests with the interview-discussion technique.

SUELTZ, BEN A. "Evaluation of Arithmetic Learnings." *National Elementary Principal* 30: 24-33; October 1950.

Illustrates newer techniques of testing.

SUELTZ, BEN A., AND BENEDICK, JOHN W. "The Need for Extending Arithmetic Learnings." *Mathematics Teacher* 43: 69-73; February 1950.

Reports testing of 2000 sixth-grade pupils; understandings and judgments as well as skill in computation and in problem solving were measured; general weaknesses were revealed in all four areas.

SUELTZ, BEN A.; BOYNTON, HOLMES; AND SAUBLE, IRENE. "The Measurement of Understanding in Elementary-School Mathematics." *The Measurement of Understanding*. Forty-Fifth Yearbook, Part I, National Society for the Study of Education. Chicago: University of Chicago Press, 1946. Chapter 7, p. 138-56.

Evaluation by (1) written tests and (2) observation, discussion and interview; emphasis on detecting understanding and meaning.

WRIGHTSTONE, J. WAYNE. "Constructing Tests of Mathematical Concepts for Young Children." *Arithmetic Teacher* 3: 81-84, 108; April 1956.

Tests designed for Grades 1 and 2; description of construction, validation, norms and reliability.

11. NUMERATION: ADDITION AND SUBTRACTION OF WHOLE NUMBERS

BROWNELL, WILLIAM A. "An Experiment on 'Borrowing' in Third-Grade Arithmetic." *Journal of Educational Research* 41: 161-71; November 1947.

Study of 328 third-grade pupils; results confirmed superiority of the equal additions method over the decomposition method when both are taught mechanically, but favored the teaching of "borrowing" by decomposition if taught rationally.

- BROWNELL, WILLIAM A., AND MOSER, HAROLD E. *Meaningful versus Mechanical Learning: A Study in Grade III Subtraction*. Research Studies in Education, No. 8. Durham, N. C.: Duke University Press, 1949. 207 p.

An elaborate study of the teaching of subtraction by two methods: (1) with emphasis on meaningful approach, and (2) with emphasis on a mechanical approach.

- DAWSON, DANIEL T. "Number Grouping as a Function of Complexity." *Elementary School Journal* 54: 35-42; September 1953.

Study of first-grade pupils' comprehension of number as related to the nature of the field of perception.

- FLOURNOY, FRANCES. "The Controversy Regarding the Teaching of Higher-Decade Addition." *Arithmetic Teacher* 3: 170-73, 176; October 1956.

Survey of the literature and textbooks with respect to the introduction and development of higher-decade addition, with particular reference to grade 3.

- GIBB, E. GLENADINE. "Children's Thinking in the Process of Subtraction." *Journal of Experimental Education* 25: 71-80; September 1956.

A study of the thinking done by second-grade children when solving problems involving three types of questions requiring subtraction.

- HIGHTOWER, HOWARD W. "Effect of Instructional Procedures on Achievement in Fundamental Operations in Arithmetic." *Educational Administration and Supervision* 40: 336-48; October 1954.

Summary of selected research studies on certain aspects of the relative merits of different methods of addition and subtraction.

- HOLMES, DARRELL. "An Experiment in Learning Number Systems." *Educational Research Bulletin* 28: 100-104, 111-12; April 1949.

Findings show that seventh-grade children can learn different number systems, and in so doing, come to a better understanding of the decimal system.

JOHNSON, DONOVAN A. "A Unit on Our Number System." *School Science and Mathematics* 52: 556-61; October 1952.

A list of 21 specific learning activities designed to help pupils understand number relationships.

RHEINS, GLADYS, B., AND RHEINS, JOEL J. "A Comparison of Two Methods of Compound Subtraction: The Decomposition Method and the Equal Additions Method." *Arithmetic Teacher* 2: 63-69; October 1955.

Comparative study using 35 pairs of students; five years after the process was taught, evidence points to the decomposition method as superior for the less intelligent pupils.

SWENSON, ESTHER J. "Difficulty Ratings of Addition Facts as Related to Learning Method." *Journal of Educational Research* 38: 81-85; October 1944.

A study of 332 second-grade pupils concludes that the comparative difficulty of addition facts is affected somewhat by the method of teaching used.

VAN ENGEN, HENRY. "Place Value and the Number System." *Arithmetic* 1947. Supplementary Educational Monographs, No. 63. Chicago: University of Chicago Press, 1947. p. 59-73.

The thesis that to rationalize any arithmetical process, an adequate understanding of the number system is essential.

WEAVER, J. FRED. "Whither Research on Compound Subtraction?" *Arithmetic Teacher* 3: 17-20; February 1956.

Review of some of the literature; raises some questions for further research.

WILBURN, D. BANKS. "Learning to Use a Ten in Subtraction." *Elementary School Journal* 47: 461-66; April 1947.

Study of 291 third-grade pupils; after 6 weeks of teaching most of them could use a ten meaningfully.

12. MULTIPLICATION AND DIVISION OF WHOLE NUMBERS

BENZ, HARRY E. "Two-Digit Divisors Ending in 4, 5, or 6." *Arithmetic Teacher* 3: 187-91; November 1956.

Presents the one-rule vs. the two-rule method for estimating quotient figures.

- BROWNELL, WILLIAM A. "Arithmetical Readiness as a Practical Classroom Concept." *Elementary School Journal* 52: 15-22; September 1951.

Study of the readiness of fifth-grade children for division by two-place divisors at the time instruction in the topic began.

- BROWNELL, WILLIAM A. "The Effects of Practicing a Complex Arithmetical Skill upon Proficiency in Its Constituent Skills." *Journal of Educational Psychology* 44: 65-81; February 1953.

An extension of a previous study on readiness for division with two-place divisors.

- BROWNELL, WILLIAM A., AND CARPER, DORIS V. *Learning the Multiplication Combinations*. Research Studies in Education, No. 7. Durham, N. C.: Duke University Press, 1943. 177 p.

An analysis of the learning process, based upon results of group test scores as well as individual interviews.

- DAWSON, DANIEL T., AND RUDDALL, ARDEN K. "An Experimental Approach to the Division Idea." *Arithmetic Teacher* 2: 6-9; February 1955.

Study of two methods of division used at fourth-grade level.

- FULLER, KENNETH G. *An Experimental Study of Two Methods of Long Division*. Contributions to Education, No. 951. New York: Teachers College, Columbia University, 1949. 76 p.

One group used the increase-by-one method while the other group used a table of multiples of the divisor up to nine times the divisor as a source for quotient figures.

- GROSSNICKLE, FOSTER E. "How to Find the Position of the Decimal Point in the Quotient." *Elementary School Journal* 52: 452-57; April 1952.

Relative merits of three methods of determining the position of the decimal point: (1) caret method; (2) multiplying dividend and divisor by a power of 10; (3) subtraction rule.

- GUNDERSON, AGNES G. "Thought-Patterns of Young Children in Learning Multiplication and Division." *Elementary School Journal* 55: 453-61; April 1955.

Study of 24 second-grade pupils; interview method; concludes that second grade is not too early to begin the study of division by means of concrete objects.

HARTUNG, MAURICE L. "Estimating the Quotient in Division." *Arithmetic Teacher* 4: 100-11; April 1957.

Analysis of the research on this problem; favors the "round up" method, at least for the early stages of instruction; bibliography.

KARSTENS, HARRY. "Estimating the Quotient Figure." *Journal of Educational Research* 38: 522-28; March 1945.

Recommends using the increase-by-one rule for all two-place divisors ending in 5; gives supporting data.

KARSTENS, HARRY. "Checking the Estimate in Long Division." *Journal of Educational Research* 40: 52-56; September 1946.

A method of multiplying the first two figures of the divisor by the estimated quotient figure and comparing with the appropriate part of the dividend.

MORTON, ROBERT L. "Estimating Quotient Figures When Dividing by Two-Place Numbers." *Elementary School Journal* 48: 141-48; November 1947.

The relative merits of the apparent method and the increase-by-one method.

MOSER, HAROLD E. "Two Procedures for Estimating Quotient Figures when Dividing by Two-Place Numbers." *Elementary School Journal* 49: 516-22; May-June 1949.

Another discussion of the one-rule and the two-rule methods.

OSBURN, W. J. "Division by Dichotomy as Applied to the Estimation of Quotient Figures." *Elementary School Journal* 50: 326-30; February 1950.

A theoretical discussion of the one-rule or two-rule controversy concerning estimation of quotient figures in long division.

SMITH, ROLLAND R. "Meaningful Division." *Mathematics Teacher* 43: 12-18; January 1950.

The presentation of division as a series of subtractions at various grade levels.

VAN ENGEN, HENRY, AND GIBB, E. GLENADINE. *General Mental Functions Associated with Division*. Education Service Studies, No. 2. Cedar Falls, Iowa: Iowa State Teachers College, 1956. 181 p.

Two teaching techniques applied to division with whole numbers at

fourth-grade level: (1) a "general-ideas" approach, and (2) a "unit-skills" approach.

13. FRACTIONS, DECIMALS, AND PERCENTS

CHRISTOFFERSON, H. C. "Division by a Fraction Made Meaningful." *Mathematics Teacher* 41: 32-35; January 1948.

Seven methods of explaining division by a fraction so as to make the process meaningful, sensible and simple.

GROSSNICKLE, FOSTER E. "Kinds of Errors in Division of Decimals and their Constancy." *Journal of Educational Research* 37: 110-17; October 1943.

Analysis of errors based on an examination of 100 random papers covering a range of grades; errors resulting from placement of quotient accounted for 40 percent of all errors.

GUILER, WALTER S. "Difficulties in Decimals Encountered by Ninth-Grade Pupils." *Elementary School Journal* 46: 384-93; March 1946.

Statistical analysis of the types of errors encountered; implications and recommendations.

GUILER, WALTER S. "Difficulties in Fractions Encountered by Ninth-Grade Pupils." *Elementary School Journal* 46: 146-56; November 1945.

Statistical analysis of the types of errors encountered; implications and recommendations.

GUILER, WALTER S. "Difficulties in Percentage Encountered by Ninth-Grade Pupils." *Elementary School Journal* 46: 563-73; June 1946.

Statistical analysis of the types of errors encountered; implications and recommendations.

JOHNSON, JOHN T. "Are Ratios Fractions?" *Elementary School Journal* 48: 374-78; March 1948.

The relation of the ratio concept to the fraction concept, with suggestions for teaching.

JOHNSON, JOHN T. "Decimal Versus Common Fractions." *Arithmetic Teacher* 3: 201-203, 206; November 1956.

Discussion of previous studies with fifth- and sixth-grade children using addition and subtraction with fractions.

- LATINO, JOSEPH J. "Take the Folly Out of Fractions." *Arithmetic Teacher* 2: 113-18; November 1955.

A suggested approach for beginning the study of fractions by eliminating unnecessary terminology and making greater use of concrete materials.

- MULHOLLAND, VERNIE. "Fifth-Grade Children Discover Fractions." *School Science and Mathematics* 54: 13-30; January 1954.

An investigation of the hypothesis that children can and do make discoveries of mathematical relationships on their own.

- POTTER, MARY A. "Corralling the Wandering Decimal Point." *Mathematics Teacher* 40: 51-57; February 1947.

Points out the disadvantages of shifting or moving the decimal point mechanically.

- RAMHARTER, HAZEL K., AND JOHNSON, HARRY C. "Methods of Attack Used by 'Good' and 'Poor' Achievers in Attempting to Correct Errors in Six Types of Subtraction Involving Fractions." *Journal of Educational Research* 42: 586-97; April 1949.

Results showed that "good" achievers had greater insight and used greater ingenuity than "poor" achievers.

- SAUBLE, IRENE. "Teaching Fractions, Decimals, and Per Cent: Practical Applications." *Arithmetic 1947*. Supplementary Educational Monographs, No. 63. Chicago: University of Chicago Press, 1947. p. 33-48.

A detailed follow-up of Thiele's article in the same monograph (see below).

- THIELE, CARL LOUIS. "Teaching Common and Decimal Fractions and Per Cent: General Issues." *Arithmetic 1947*. Supplementary Educational Monographs, No. 63. Chicago: University of Chicago Press, 1947. p. 23-32.

Overview of the methodology of these topics, with specific suggestions.

- TRIMBLE, HAROLD C. "Fractions Are Ratios, Too." *Elementary School Journal* 49: 285-91; January 1949.

The concept of the fraction as a ratio can be developed concretely as a basis for understanding.

14. PREPARATION OF TEACHERS

BOYER, LEE EMERSON. "Preparation of Elementary Arithmetic Teachers." *Emerging Practices in Mathematics Education*. Twenty-Second Yearbook. Washington, D. C.: National Council of Teachers of Mathematics, a department of the National Education Association, 1954. p. 172-80.

Emphasis is upon the content of arithmetic—mathematical, historical and cultural.

DUTTON, WILBUR H. "Attitudes of Prospective Teachers Toward Arithmetic." *Elementary School Journal* 52: 84-90; October 1951.

DUTTON, WILBUR H. "Measuring Attitudes Toward Arithmetic." *Elementary School Journal* 55: 24-31; September 1954.

Both papers present objective data concerning attitudes of prospective teachers, chiefly undergraduates at the college or university level.

GLENNON, VINCENT J. "A Study in Needed Redirection in the Preparation of Teachers of Arithmetic." *Mathematics Teacher* 42: 389-96; December 1949.

Findings and recommendations based upon results of a test on basic mathematical understandings administered to 476 prospective teachers and teachers-in-service.

GROSSNICKLE, FOSTER E. "The Training of Teachers of Arithmetic." *The Teaching of Arithmetic*. Fiftieth Yearbook, Part II, National Society for the Study of Education. Chicago: University of Chicago Press, 1951. Chapter 11, p. 203-31.

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